# Characterization of the chaos-hyperchaos transition based on return times

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We discuss the problem of the detection of hyperchaotic oscillations in coupled nonlinear systems when the available information about this complex dynamical regime is very limited. We demonstrate the ability of diagnosing the chaos-hyperchaos transition from return times into a Poincaré section and show that an appropriate selection of the secant plane allows a correct estimation of two positive Lyapunov exponents (LEs) from even a single sequence of return times. We propose a generalized approach for extracting dynamics from point processes that allows avoiding spurious identification of the dynamical regime caused by artifacts. The estimated LEs are nearly close to their expected values if the second positive LE is essentially different from the largest one. If both exponents become nearly close, an underestimation of the second LE may be obtained. Nevertheless, distinctions between chaotic and hyperchaotic regimes are clearly possible.

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## I. INTRODUCTION

A chaos-hyperchaos transition in complex nonlinear systems is relatively easy to identify if the mathematical model of the considered system is known. Based on the standard approaches for determining the Lyapunov exponents [1,2], parameter values associated with the appearance of two or more positive LEs can then be estimated in a quite accurate way. The Lyapunov spectrum ($\lambda_1$, $\lambda_2$, ..., $\lambda_k$) provides a clear quantification of the increased complexity associated with a hyperchaotic state as compared with chaotic oscillations described by a single positive LE. The higher complexity of this state is caused by bifurcations of unstable periodic orbits embedded in the chaotic attractor [3]. However, the characterization of this increased complexity becomes more difficult when the dynamical equations are unknown as is often the case in natural systems as in physiology or earth science.

Recently, an effective method for detection of chaos-hyperchaos transitions based on the recurrence plots (RPs) was proposed [4] that can be used even for relatively short time series, and its application for model and experimental systems was verified [5]. The method [4,5] distinguishes between the discussed dynamical regimes based on several measures introduced for RPs [6]. Although the possibility of diagnosing hyperchaos from time series was demonstrated in many studies, the question of how much information is required for a correct identification of complex dynamical regimes with two or more positive LEs still remains open.

Unlike the method [4], in this work we consider an idea of restoring the averaged instantaneous frequency of complex oscillations to estimate two largest LEs using very limited information about complex dynamical regimes. We discuss the problem of analyzing hyperchaotic dynamics in coupled nonlinear systems based on return times into a Poincaré section that has a relation to the more general problem of quantifying systems dynamics from point processes [7] and, in particular, the reconstruction of dynamical systems [8]. During the last few decades this problem was discussed for several types of simple neuron models, including threshold crossing (TC) and integrate-and-fire (IF) models [9]. The TC model describes the generation of spikes when an input signal $x(t)$ crosses some threshold value $\Theta$ in one direction, while the IF model produces corresponding events if the integral from an input signal $x(t)$ reaches a selected constant value $\theta$, and this integral is set to zero after the generation of each spike. There is an analogy between interspike intervals of the TC model and return times into the Poincaré section introduced as $x(t) = \Theta$, if $x(t)$ is the phase space coordinate.

An approach for computing dynamical characteristics from return times was proposed in Ref. [10], and it was shown that the averaged instantaneous frequency of complex oscillations restored from the sequence of return times enables us to estimate the largest LE even when the intersections of the plane $x(t) = \Theta$ do not occur during some rotations of the phase space trajectory [11]. A necessary condition is that the mean return time is less than the prediction time for the analyzed dynamical regime [11,12]. However, the approach [10] can lead to spurious identification of the dynamical regime caused by artifacts. Due to this, we consider in this paper a generalized approach being able to correctly detect chaotic and hyperchaotic oscillations. We show that this generalized approach is able to diagnose the chaos-hyperchaos transition using a single sequence of return times into a Poincaré section if the latter is appropriately selected. We demonstrate that the estimated LEs are nearly close to the values computed using the mathematical model of the considered system when the analyzed dynamical regime is quantified by essentially different values of $\lambda_1$ and $\lambda_2$. When $\lambda_1 \simeq \lambda_2$, the detection of the chaos-hyperchaos transition is also provided despite an underestimated value of the second LE.

## II. COMPUTING LYAPUNOV EXPONENTS FROM RETURN TIMES

The main idea underlying the approach [10] consists of the following steps. Let us consider a chaotic system with $x(t)$

 

being one of the phase space coordinates and introduce the Poincaré section as $x(t) = \Theta$. If $T_i$ are the times of crossings of this section in one direction, and $I_i$ are return times, $I_i = T_{i+1} - T_i$, then we may estimate the points

$$\omega(T_i) = \frac{2\pi}{I_i} \tag{1}$$

representing the values of the instantaneous frequency of chaotic oscillations $x(t)$ averaged during a return time $I_i$. The values $\omega(T_i)$ are qualitatively treated as the samples of the instantaneous frequency introduced via the Hilbert transform that are obtained using an averaging method with a varying window [10]. Moreover, these samples are known only at discrete time moments $T_i$. In order to analyze dynamical properties of the considered regime, we need to introduce a constant time step between data points. The latter can be done by interpolation of the samples $\omega(T_i)$ with a smooth function, e.g., a cubic spline. Although we do not exactly reproduce the true dependence of the instantaneous frequency, the time series $\omega(j\Delta t)$ obtained in such a way allows approximate reconstruction of the chaotic attractor with the standard delay method [13] and, therefore, estimation of dynamical and geometrical properties of the reconstructed attractor. A good quality of determining Lyapunov exponents with this approach (with an error of about 10%) was demonstrated in Refs. [10,11] using different chaotic oscillators.

In order to compute LEs from the interpolated time series $\omega(j\Delta t)$ we used the method for LEs estimation proposed by Wolf *et al.* [14]. Although the approach [10] provides a possibility to estimate two LEs, its application to a single sequence of return times related to hyperchaotic oscillations was not studied.

Aiming to quantify dynamical features of hyperchaotic regimes based on return time sequences, we consider the following paradigmatic model of two coupled Rössler oscillators [15]:

$$\begin{aligned}
\frac{dx_1}{dt} &= -w_1 y_1 - z_1 + \gamma(x_2 - x_1),\\
\frac{dy_1}{dt} &= w_1 x_1 + a y_1,\\
\frac{dz_1}{dt} &= b + z_1(x_1 - c),\\
\frac{dx_2}{dt} &= -w_2 y_2 - z_2 + \gamma(x_1 - x_2),\\
\frac{dy_2}{dt} &= w_2 x_2 + a y_2,\\
\frac{dz_2}{dt} &= b + z_2(x_2 - c).
\end{aligned} \tag{2}$$

Here the parameters $a$, $b$, and $c$ govern the dynamics of each subsystem, and $\gamma$ characterizes the coupling strength. The basic frequencies $w_1 = w_0 + \Delta$ and $w_2 = w_0 - \Delta$ have a small mismatch $\Delta$ that provides distinctions of the considered dynamics of both individual oscillators. In this study, the following parameter set is chosen [16]: $a = 0.15$, $b = 0.2$, $\gamma = 0.02$, $w_0 = 1.0$, $\Delta = 0.0093$. The parameter $c$ varies in the range [6.8,8.0] including regions of both, chaotic, and hyperchaotic oscillations. The transition to hyperchaos

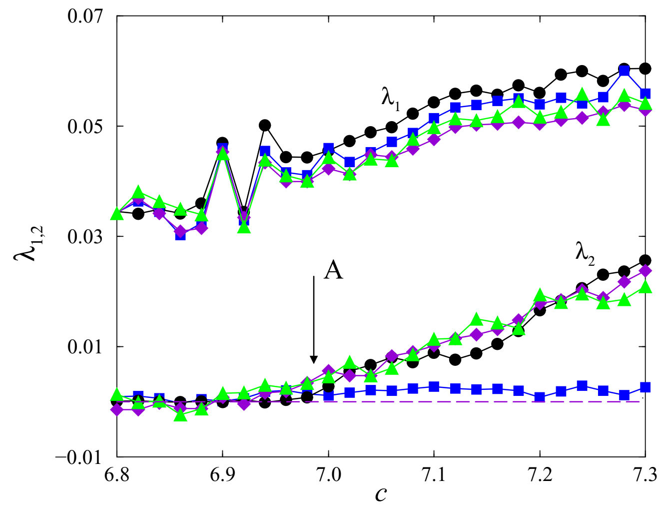


FIG. 1. (Color online) Dependencies of $\lambda_1$ and $\lambda_2$ versus the parameter $c$ of the coupled Rössler oscillators (2). Circles indicate "true" values estimated with the method [1]. Squares are related to LEs estimated from the sequences of return times into the plane $x_1 = 0$. Diamonds denote LEs computed from two sequences of return times into the planes $x_1 = 0$ and $x_2 = 0$, when the restored attractor contains delayed coordinates related to the averaged instantaneous frequencies for both subsystems. Triangles are related to LEs estimated from a single sequence of return times into the plane $x_2 + y_1 = 0$. The arrow marked by A shows the chaos-hyperchaos transition.

is illustrated in Fig. 1, where the dependencies of the two largest LEs versus the parameter $c$ are illustrated. The first LE is positive in the whole considered range of $c$ verifying the presence of exponential instability of trajectories associated with the chaotic oscillations. The second LE $\lambda_2 > 0$ for $c > 7.0$ quantifying the presence of a more complicated dynamical regime. Circles in Fig. 1 correspond to "true" values of LEs estimated using the standard approach [1], i.e., based on the known mathematical model (2). The approach [1] assumes that LEs are estimated by consideration of the long-term evolution of the axes of an infinitesimal sphere of initial states using the so-called fiducial trajectory associated with the center of the sphere and the Gram-Schmidt reorthonormalization. In order to reach the convergence of LEs, the fiducial trajectory may include thousands or millions orbital periods (mean return times). The obtained values are used for comparison with the estimations performed from point processes.

In contrast to the former estimation we now analyze a time series consisting of only 2000 samples. Squares in Fig. 1 mark the values of two LEs computed from the sequences of return times into the plane $x_1 = 0$. Note that a consideration of time intervals between intersections of this plane in one direction allows obtaining of only the largest positive LE estimated within the approach described in Refs. [10,11], while a zero value is obtained for the second LE, which is different from the expected exponent $\lambda_2$. We can, therefore, conclude that a single sequence of return times into the plane $x_1 = 0$ is not enough for diagnostics of hyperchaos. The latter circumstance is caused by inappropriate selection of the secant plane, which does not provide enough information about the whole dynamics of coupled oscillators. The analogous situation is observed for the secant plane introduced as $x_2 = 0$. Thus, consideration of a secant plane introduced for only one subsystem of the coupled oscillators (2) does not allow a correct characterization of hyperchaotic dynamics.

Aiming to correctly estimate the two largest (positive) LEs in the hyperchaotic regime, the delay reconstruction [13] based on two time series can be performed. In the latter case, the reconstructed attractor contains half of the coordinates introduced by the delayed time series (in our case, the averaged instantaneous frequency) for one subsystem and another half of the coordinates is obtained analogously for the second subsystem. This modification of the reconstruction stage provides a good estimation of two positive LEs (Fig. 1, diamonds), and they are both relatively close to the expected values. Thus, the chaos-hyperchaos transition is clearly and correctly identified verifying advantages of the reconstruction based on two time series. In this study we consider quite short sequences of return times (2000 samples) in order to illustrate the method's performance at the condition when the limitation of available information is caused not only by using point processes, but also by a rather short observation time. There remains, however, the following question: Are two sequences of return times indeed necessary for LEs estimation, or is it possible to use a single sequence for this purpose with an appropriate selection of the secant plane?

In order to quantify hyperchaotic dynamics based on a single sequence of return times, we need to introduce the secant plane in such a way that it accounts for the dynamics of both subsystems simultaneously and in more detail. Triangles in Fig. 1 show the result for the secant plane introduced as $x_2 + y_1 = 0$. In this case we use one sequence of return times that is enough for a correct estimation of both positive LEs quantifying the hyperchaotic regime. From the viewpoint of the TC model, this case is related to the thresholding of a summary signal measured at the output of both subsystems. Thus, time intervals between zero crossings by a single (a summary) signal provide a possibility to correctly identify numerical measures of hyperchaotic dynamics.

To avoid spurious identification of dynamical regimes, we substantially extended the method [10] by including several important items:

First, the output sequence $I_i$ should be checked for possible artifacts including the generation of additional spikes (i.e., when a summation of two phase space coordinates leads to the appearance of pairs of spikes instead of single spiking events) or missing some spikes for large thresholds $\Theta$. Such artifacts often complicate the analysis of neural systems [17]. When missing of spikes occurs, the output sequence $I_i$ contains time intervals being close to the values $2\overline{T}$, $3\overline{T}$, etc., where $\overline{T}$ is the basic period of oscillations, i.e., the period corresponding to the basic frequency in the power spectrum (in the regime of the phase-coherent dynamics). In this case, samples of the averaged frequency $\omega(T_i)$ should be estimated as $\omega(T_i) = 2\pi m/I_i$, where $m$ is selected from the condition of slow changes of $\omega(t)$. The latter allows us avoiding incorrect samples of $\omega(T_i)$ caused by inappropriate choice of the secant plane. The generation of additional spikes when two nearby events occur whose sum is close to $\overline{T}$ also leads to incorrect samples of $\omega(T_i)$. Without correction of these artifacts, the estimated LEs may essentially differ from their expected values. In particular, $\lambda_2 > 0$ may be obtained in a chaotic regime instead of a zero value.

Another extention of the method [10] for sequences of $I_i$ with a broad distribution of interspike intervals consists in performing additional estimations depending on the interpolation technique (splines, polynomials, etc.). If the difference of LEs estimated for two smooth functions (different interpolation techniques) exceeds 10%, this is a sign of a high variability of the obtained results. In the latter case, computing LEs should be provided depending on the main reconstruction parameters in order to get a clear and correct characterization of the studied dynamical regime and, in particular, of chaos-hyperchaos transitions.

Let us note that computing LEs from time series using the reconstruction approach [14] is accompanied by orientation errors that have a tendency to accumulate for each sequential LE. That is why we provide estimations of only two LEs that can be computed with an appropriate quality. The discussed approach is also effective when the external noise influences the system's dynamics if noise intensity is quite small and the noise-induced additional divergence of trajectories is significantly less than the scale associated with the limit of the linear approach used to compute LEs (typically, 5%–10% of the attractor size).

## III. ROLE OF THE SECANT PLANE AND AMOUNT OF DATA

In Fig. 1 estimation of LEs from a single point process was performed for the plane $x_2 + y_1 = 0$, i.e., for the case of nearly equal contribution of both subsystems of Eq. (2). If the impact of one subsystem outperforms the contribution of another one in a summary signal, the results may essentially differ. Aiming to illustrate this circumstance, we considered zero crossings of a signal at the input of the TC model selected as $k_1 y_2(t) + k_2 x_1(t)$ or $k_1 y_1(t) + k_2 x_2(t)$, where $k_1 = \sin\alpha$, $k_2 = \cos\alpha$. The results are presented in Fig. 2. Depending on $\alpha$ we selected those secant plane for which artifacts caused by generation of additional spikes do not occur. We see that correct results for both exponents are related to the case when contributions of both subsystems are comparable ($\alpha$ is close to $\pi/4$ or $3\pi/4$). If a signal of one subsystem dominates, incorrect diagnostics of the hyperchaotic regime may occur. Note, that the largest LE is correctly estimated in a wider range of $\alpha$ as compared with $\lambda_2$. This testifies that computing the second LE represents a much more difficult procedure.

In Figs. 1 and 2 we present the case of two essentially different positive LEs. If the second LE approaches to the first LE (Fig. 3), the errors caused by the vector orientation in the reconstructed phase space at the performing reorthonormalization procedure may increase and accumulate for the second LE, if the direction of the trajectories convergence for $\lambda_1$ and $\lambda_2$ becomes closer. The problem of estimating successive exponents when they are close was also reported in Ref. [14], where it was discussed that LEs with significantly different values are easier to estimate. As a result, an underestimated second LE may obtain; i.e., the error of computing the second LE becomes larger than for the first LE (Fig. 3). Nevertheless, the presence of a hyperchaotic regime is identified, and the

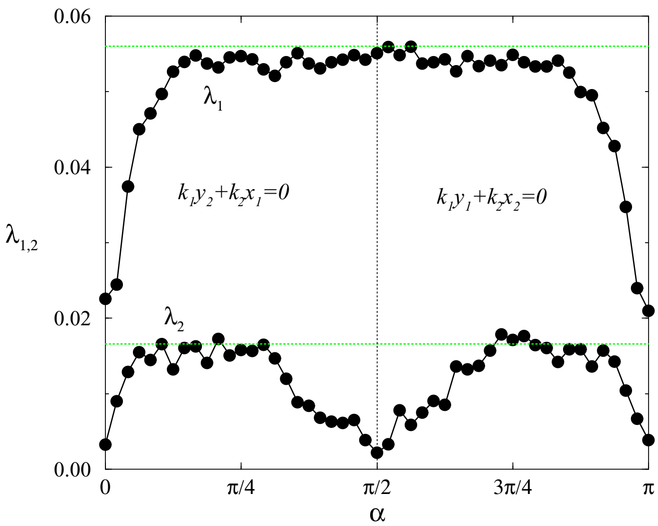


FIG. 2. (Color online) Computing two largest LEs depending from the secant plane introduced as $k_1 y_2 + k_2 x_1 = 0$ or $k_1 y_1 + k_2 x_2 = 0$, where $k_1 = \sin\alpha$, $k_2 = \cos\alpha$. Dashed lines mark "true" values of LEs.

discussed approach separates between chaotic and hyperchaotic dynamics based on a single series of return times.

In general, the generalized approach allows to use a reduced number of return times in order to provide an appropriate quantification of the hyperchaotic dynamics. Large sequences of return times decrease estimation errors and possible fluctuations of $\lambda_{1,2}$ caused by an inappropriate selection of the reconstruction parameters. Nevertheless, a characterization of the chaos-hyperchaos transition is possible for a smaller amount of data points. Figure 4 shows how the estimated values

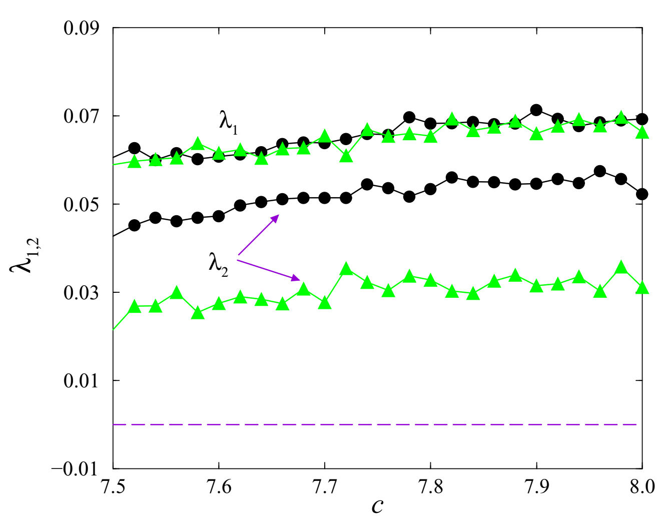


FIG. 3. (Color online) Dependencies of two largest LEs versus the parameter $c$ of the coupled Rössler oscillators (2) for the parameter ranges related to nearly close LEs. Circles indicate "true" values estimated with the method [1]. Triangles correspond to $\lambda_{1,2}$ computed from sequences of return times into the secant plane $x_2 + y_1 = 0$ (2000 samples). Note that an underestimated $\lambda_2$ is computed from point processes that is caused by orientation errors.

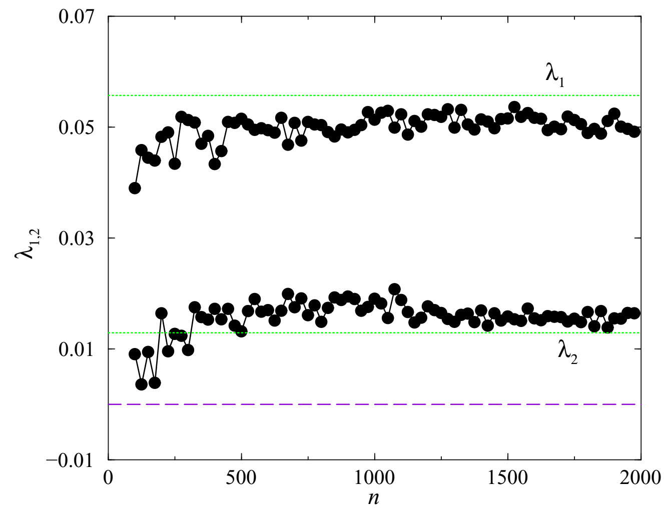


FIG. 4. (Color online) Dependencies of two largest LEs versus the number of analyzed return times for the parameter $c = 7.18$ of the coupled Rössler oscillators (2) and the secant plane $x_2 + y_1 = 0$. Dotted lines indicate "true" values of $\lambda_{1,2}$ estimated with the method [1].

of $\lambda_{1,2}$ depend on the number of return times. According to this figure, even sequences of about 500 samples allow obtaining of two largest LEs that approach the expected values. Depending on the required precision, the length of data series can be selected more accurately. Thus, in order to compute $\lambda_2$ with an absolute error less than 0.005, one needs sequences of at least 1200 return times.

## IV. CONCLUSION

In summary, we extended the approach for extracting dynamics from point processes [10,11] in order to avoid spurious identification of the dynamical regime caused by artifacts. The generalized approach is able to correctly estimate two positive LEs quantifying the dynamics of coupled nonlinear oscillators. This estimation can be performed from quite short sequences of return times for dynamical regimes quantifying by essentially different values of $\lambda_1$ and $\lambda_2$. An important circumstance is the selection of the secant plane. If the latter is selected in such a way that it accounts for the whole system dynamics, it becomes possible to obtain correct values of positive LEs and, therefore, to detect the chaos-hyperchaos transition. In the case of hyperchaotic oscillations with rather close positive LEs, characterization of the chaos-hyperchaos transition is also clearly and correctly performed despite underestimated values of $\lambda_2$.

The transition to hyperchaos was earlier studied based on the statistics of return times and the Poincareé maps in a number of publications. Thus, the work [18] established a scaling law for such transition using the projections of Poincaré maps associated with the dynamics of two coupled van der Pol oscillators with a chaotic driving. With the statistics of the Poincareé maps, different properties of the chaos-hyperchaos transition were described in Ref. [19]. A promising tool for studying this transition is the application of measures based on RPs. The works [4,5] discussed, e.g., some measures that provide clear characterization of the transition to hyperchaos. However, they typically use larger data sets as compared to the given study.


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