Lyapunov exponent corresponding to enslaved phase dynamics: Estimation from time series

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A method for the estimation of the Lyapunov exponent corresponding to enslaved phase dynamics from time series has been proposed. It is valid for both nonautonomous systems demonstrating periodic dynamics in the presence of noise and coupled chaotic oscillators and allows us to estimate precisely enough the value of this Lyapunov exponent in the supercritical region of the control parameters. The main results are illustrated with the help of the examples of the noised circle map, the nonautonomous Van der Pole oscillator in the presence of noise, and coupled chaotic Rössler systems.

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I. INTRODUCTION

Lyapunov exponents (LEs) are a powerful tool for the analysis of complex system dynamics [1–4]. In particular, they are used for identification of the transition between different regimes (e.g., from periodic and quasiperiodic oscillations to the chaotic ones [5,6] or from chaotic oscillations to the hyperchaotic ones [7]) to reveal the presence of hyperbolic attractor [3,8], as well as for the different types of synchronization detection (see, e.g., Refs. [9–13]).

To calculate the values of Lyapunov exponents effective procedures and algorithms have been proposed [14–17]. In particular, in the case when the evolution operator is known implicitly the spectrum of Lyapunov exponents can be found easily by means of standard procedures based on the numerical calculation of Lyapunov sums with the help of the system evolution operator and its linearization, with these approaches being applicable for both the system with the small number of degrees of freedom [16,17] and the spatially extended systems [4,18]. Nevertheless, sometimes (e.g., for the experimental time series analysis) it is necessary to compute Lyapunov exponents when the only one accessible characteristic is the time realization of the system under study. Several methods (see, e.g., Refs. [2,17,19–23]) allowing us to estimate the one or two highest Lyapunov exponents from time series are known. At the same time, not only is the largest Lyapunov exponent value important to characterize the system dynamics. For example, the zero Lyapunov exponent plays a significant role for different phenomena, such as for the quasiperiodic oscillations or for the different types of the synchronous motion such as the phase synchronization [24–27] or incomplete noise-induced synchronization [28].

The zero Lyapunov exponent exists necessarily in the Lyapunov exponent spectrum of the flow system characterizing the time evolution of the perturbation along the trajectory in the phase space. For two coupled flow oscillators (which possesses two zero Lyapunov exponents when the coupling strength between them is equal to zero) one of the zero Lyapunov exponents diverges from the zero value with the increase of the coupling strength due to the enslaved phase dynamics.

The zero Lyapunov exponent plays a crucial role in synchronization phenomena. In particular, the transition of such a Lyapunov exponent in the field of the negative values is known to be closely connected with the onset of the phase synchronization regime in both systems demonstrating periodic dynamics in the presence of noise and chaotic oscillators [24,29]. Moreover, the magnitude of the value of the Lyapunov exponent corresponding to the enslaved phase dynamics characterizes the degree of synchronism of the interacting coupled systems; therefore, this Lyapunov exponent value can be useful in many relevant circumstances (e.g., physical, biological, or medical) to estimate the degree of the synchronism of oscillations.

As has been mentioned above, the classical technique of Wolf *et al.* [17] allows calculating only the largest exponents, whereas, e.g., for two coupled chaotic systems the Lyapunov exponent corresponding to the enslaved phase dynamics is the fourth one. The estimation of each subsequent exponent in the Lyapunov exponent spectrum becomes more and more inaccurate. The use of the local mappings with higher-order Taylor series (contrary to the linear one) improves the quality of the Lyapunov exponent value estimation [30,31], but in this case there are several aspects such as (1) the requirement of the low level of noise, (2) the existence of some limit to the number of exponents that can be accurately determined from a given finite data set, and (3) spurious exponents being generated. Additionally, this approach is rather complicated and requires the calculations of added characteristics such as the fractal dimension of the attractor and Lyapunov direction vectors. So the estimation of the value of the Lyapunov exponent corresponding to the enslaved phase dynamics is a rather nontrivial and complicated task. Therefore, to obtain this Lyapunov exponent value for the synchronization regime different methods for the estimation of the Lyapunov exponent from time series should be developed and used. In the present paper we propose an effective method for estimation of the conditional Lyapunov exponent corresponding to the enslaved phase dynamics from time series of the nonautonomous or coupled dynamical systems.

II. METHOD DESCRIPTION

The proposed method for the estimation of the value of the Lyapunov exponent corresponding to the enslaved phase

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dynamics is based on the circle map [5,32,33] with noise

$$\phi_{n+1} = \phi_n + 2\Omega[1 - \cos(\phi_n - \psi)] - \varepsilon + \xi_n, \quad \text{mod} \quad 2\pi, \tag{1}$$

where ϕ_n plays the role of the phase difference, ξ_n is the noise term, and Ω and ε are the control parameters.

The circle map is the classical model used frequently to study nonlinear phenomena [34–36] including synchronization [29] and phase locking [37,38]. The circle map describes very precisely the behavior of the periodically forced weakly nonlinear isochronous oscillator in the vicinity of the synchronization threshold in the case of the small parameter mistuning, since it may be considered as the discretization of Adler's equation [39], which, in turn, may be easily obtained from the truncated equation [40,41] deduced in the framework of the complex amplitude method. In the absence of noise ($D = 0$) the tangential bifurcation takes place at $\varepsilon_c = 0$ in (1) when the stable and unstable points

$$\phi_{s,u} = \psi \mp \arccos\left(1 - \frac{\varepsilon}{2\Omega}\right) + 2\pi n, \quad n \in \mathbb{Z} \tag{2}$$

touch each other in $\phi_c = \psi$ and disappear, which corresponds to the synchronization threshold.

The very same model is also applicable to the chaotic (or noised) oscillators with the phase coherent attractors and small frequency detuning [29,42], since (1) under this condition the chaotic oscillators may be modeled by a noised periodic oscillator [13,29,43] and (2) the mechanisms of the phase synchronization destruction and phenomena observed in the vicinity of the synchronization boundary for the chaotic and noised oscillators remain the same (but, masked by the irregular dynamics) as in the case of the periodically forced weakly nonlinear isochronous oscillator [27,29,42]. Therefore, to estimate the value of the Lyapunov exponent corresponding to the enslaved phase dynamics for the chaotic (or noised oscillators) the circle map with the added noise term modeling the chaotic dynamics (or random perturbations) is the very appropriate model system.

Since the Lyapunov exponent under study is very close to zero below the synchronization threshold [26,29], there is a reason to consider its value only in the phase-locking regime when the phase synchronization is observed and the Lyapunov exponent value is negative. As far as the the model system (1) is concerned, the positive values of the ε parameter with the localization of ϕ_n around the stable fixed point ϕ_s correspond to this type of dynamics of the driven flow oscillators.

When the values of the ϕ_n variable are localized around the stable fixed point, the circle map may be linearized in the neighborhood of ϕ_c to get the quadratic map

$$x_{n+1} = f(x_n) + \xi_n = x_n + \Omega x_n^2 - \varepsilon + \xi_n \tag{3}$$

describing perfectly the dynamics of the circle map around the bifurcation point (here $x = \phi - \psi$).

Due to the one-dimensional character of the system (3) its Lyapunov exponent can be found as

$$\Lambda_0(\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|, \tag{4}$$

where x_n is a time realization of the system (3), and $f'(x) = 1 + 2\Omega x$ is the derivative from the evolution operator calculated analytically.

Taking into account the ergodicity of the considered process, the time averaging can be substituted by the ensemble one. In this case

$$\Lambda_0(\varepsilon) = \int_{-\infty}^{+\infty} \rho_i(x) \ln |f'(x)|\, dx, \tag{5}$$

where $\rho_i(x)$ is the invariant probability density of the x variable. In Ref. [42] we have shown that in the supercritical region of the control parameter ε corresponding to the synchronization regime of the flow systems the probability density obeys the relation

$$\rho(x) = A \exp\left[-\frac{2}{D}\left(\varepsilon x - \frac{\Omega x^3}{3}\right)\right], \quad x \leqslant \sqrt{\frac{\varepsilon}{\Omega}}, \tag{6}$$

where A is the normalization factor and D is the noise variance. Equation (6) obtained in Ref. [42] was deduced under the assumption that noise term in (3) is δ-correlated Gaussian noise [$\langle \xi_n \rangle = 0$, $\langle \xi_n \xi_m \rangle = D\delta(n - m)$], but many recent works (e.g., Refs. [42,44,45]) give evidence that this assumption is a rather redundant condition and the obtained relation may be applied for the description of the synchronization phenomenon in the presence not only of Gaussian noise, but also the other types of random processes (e.g., with the uniform probability distribution) as well as the deterministic chaotic perturbations.

The relation (6) describing the probability density $\rho(x)$ is applicable for $x \leqslant \sqrt{\varepsilon/\Omega}$ and reaches its maximum at point

$$x_{\max} = -\sqrt{\frac{\varepsilon}{\Omega}}, \tag{7}$$

which may be considered also as the linearization of Eq. (2) obtained for the stable fixed point.

The phase-locking regime in the flow oscillators is modeled by the discrete map (3) with $\varepsilon/D^{2/3} \geqslant 1$ when the values of the x_n variable are localized around the stable fixed point $x_{\max}$ and the probability density $\rho(x)$ decreases rather rapidly. As a consequence, for $x < x_1^* = -2\sqrt{\varepsilon/\Omega}$ and $x > x_2^* = \sqrt{\varepsilon/\Omega}$ the value of $\rho(x)$ may be estimated as zero, and, therefore, the normalization factor A in (6) may be obtained with the help of the normalization condition written as

$$\int_{x_1^*}^{x_2^*} \rho(x)\, dx = 1, \tag{8}$$

whereas the value of the Lyapunov exponent under study can be estimated by the formula

$$\Lambda_0(\varepsilon) = \int_{x_1^*}^{x_2^*} \rho(x) \ln |1 + 2\Omega x|\, dx. \tag{9}$$

In fact, for the experimental time series the probability density decreases more rapidly; therefore, for practical purposes the boundary values of the probability distribution in (8) and (9) of phase differences may be taken as $x_1 > x_1^*$ and $x_2 < x_2^*$ values.

Having analyzed all assumptions made above, we can say that the proposed method may be used to estimate the value of the Lyapunov exponent corresponding to the enslaved phase dynamics for the chaotic or noised oscillators with small frequency detuning and the phase coherent attractors being in the phase-locking regime. To estimate the value of the Lyapunov exponent, only the parameters ε, Ω, and D of the probability distribution are required. These parameters may be obtained with the help of the approximation of the experimentally (or numerically) obtained distribution of the phase differences $\rho(\Delta\varphi)$ of the interacting systems by the theoretical distribution (6), e.g., with the help of the least square technique [46].

In general, to estimate the value of the Lyapunov exponent corresponding to the enslaved phase dynamics for the time series obtained experimentally or numerically the following steps must be done:

(1) Calculation of the phase difference quantity $\Delta\varphi(t)$ sampled in time, i.e., $\Delta\varphi_n = \Delta\varphi(t_n)$, $n = 0,1,\ldots,N$. The discretization appears in the natural way, since the initial time series are already discretized. The phase difference $\Delta\varphi(t)$ may be found as the difference between the instantaneous phases of the interacting oscillators $\varphi_{1,2}(t)$, which, in turn, can be introduced as the angles on the phase plane projections [25,43] or with the help of the continuous wavelet transform [47–49]. If the initial time series are univariate, to introduce the instantaneous phase as the angle on the phase plane projection the delay-coordinate embedding method [50] should be also used. Alternatively, the phase difference $\Delta\varphi(t)$ may be also obtained with the help of the rotating plane approach [51–53].

(2) Plotting the obtained data on the plane $(\Delta\varphi_{n+1},\Delta\varphi_n)$ and approximation of the obtained points by the quadratic function $a_0 + a_1x + a_2x^2$ (e.g., with the help of the least square method) with the linear transformation of the phase difference variable $\Delta\varphi$ in the form $x_n = \Delta\varphi_n - \psi$ (where $\psi = \text{const}$) to get the coefficient of the approximation to be $a_1 = 1$. This step is required for the simplified Eq. (3) to be used instead of Eq. (1).

(3) Calculation of the probability density $\rho(x)$ of the obtained x variable.

(4) Finding the values of parameters Ω, D, and ε and coefficient A by means of approximation of numerically obtained distribution by the regularity (6) with normalization condition (8).

(5) Finally, the estimation of the value of the Lyapunov exponent corresponding to the enslaved phase dynamics with the help of Eq. (9).

III. ESTIMATION OF VALUE OF THE LYAPUNOV EXPONENT CORRESPONDING TO ENSLAVED PHASE DYNAMICS IN MODEL SYSTEMS

In this section we consider the estimation of the value of the Lyapunov exponent corresponding to the enslaved phase dynamics for the several model systems both with discrete and continuous time. As such test systems we have selected (1) the quadratic and circle maps, (2) driven Van der Pol oscillator with noise, and (3) two coupled chaotic Rössler oscillators.

A. Quadratic and circle maps

1. Quadratic map

We start the consideration from the discrete maps being the base model of the method developed in the previous section.

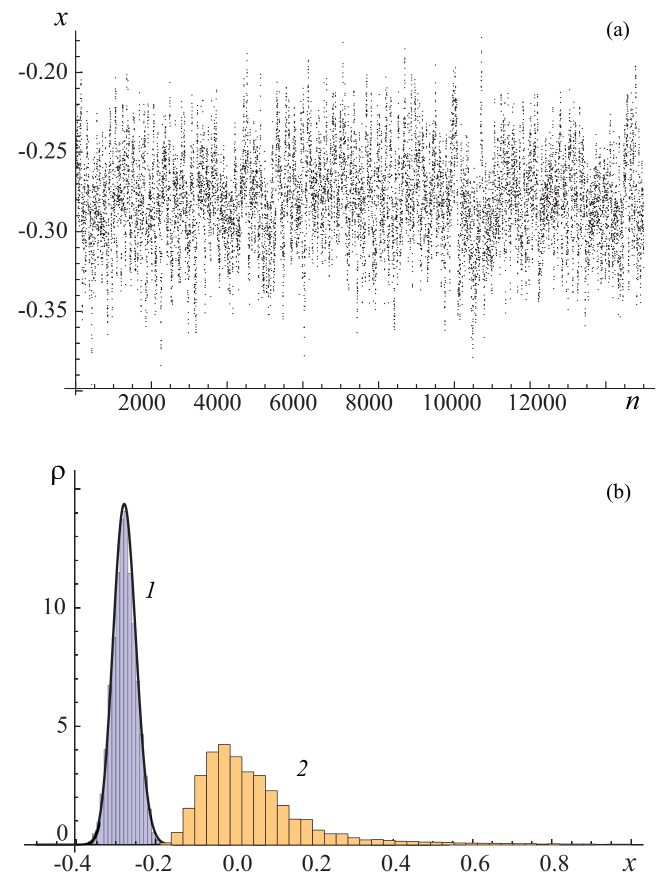

FIG. 1. (Color online) (a) Time realization of quadratic map (3) for $\Omega = 0.1$, $\varepsilon = 0.008$, $D = 0.0001$; (b) probability density $\rho(x)$ (histogram 1) obtained by time realization x_n and its approximation by regularity (6) at $A = 1.07 \times 10^{-13}$, $\Omega = 0.111\,124$, $\varepsilon = 0.0087$, $D = 0.0001$ (solid line), and analogous probability density $\rho(x)$ (histogram 2) obtained by time realization x_n for $\Omega = 0.1$, $\varepsilon = 0.0$, $D = 0.0001$.

For the quadratic map (3) in the absence of noise the tangential bifurcation takes place at $\varepsilon_c = 0$, which results in the transition of the initially zero Lyapunov exponent to the negative values. Later we fix the values of the control parameters as $\Omega = 0.1$, $\varepsilon = 0.008$, $D = 0.0001$ (which correspond to the supercritical region $\varepsilon > \varepsilon_c$) and analyze the behavior of such a Lyapunov exponent.

The idea of the proposed method is illustrated in Fig. 1. Since we deal initially with the quadratic map, steps 1 and 2 should be omitted and the starting point in this case is step 3. In Fig. 1 the time realization x_n of system (3) for the selected values of the control parameters (a) and probability density $\rho(x)$ obtained by time realization x_n (histogram 1) as well as its approximation by the regularity (6) with $A = 1.07 \times 10^{-13}$, $\Omega = 0.111\,124$, $\varepsilon = 0.0087$, $D = 0.0001$ (solid line) (b) are shown.

The values of the approximation parameters have been defined in the following way. Parameter D has been computed as an effective phase diffusion coefficient [54]. The relation between parameters A, Ω, and ε has been found from the condition of the coincidence of maxima of numerically obtained probability density and regularity (6), i.e., from condition (7), which results in the following connection between them:

$\varepsilon = 0.0784\Omega$, $A = 14.2987\exp(-292.693\Omega)$. The search for the Ω parameter has been performed by least square method. It is clearly seen that the parameters of approximation are close to the initial values of the control parameters Ω and ε. Substituting them into relation (9) and choosing $x_1 = -0.5$, $x_2 = 0$ we obtain a value of $\Lambda_0 = -0.065$, which is in a good agreement with the value of Lyapunov exponent corresponding to enslaved phase dynamics computed numerically by formula (4).

Fig. 1(b) (histogram 2) obtained for $\varepsilon = 0$ illustrates why the proposed method becomes worse approaching the transition point ε_c. One can see that in this case the probability distribution of the variable x_n is not localized in the neighbourhood of the certain fixed point. Since the assumptions used for the derivation of Eq. (6) are not met, expression (6) does not allow us to get the acceptable approximation of the probability distribution $\rho(x)$ [see histogram 2 in Fig. 1(b)], and, as a consequence, the estimated value of the Lyapunov exponent under study is not correct before and in proximity to the transition point.

2. *Circle maps*

Let us apply the proposed approach to the other systems and consider the circle map (1) being the more generalized model of the synchronization phenomenon. The control parameter values and characteristics of the noise signal we have chosen are the same as in the case of the quadratic map (3) considered above, $\psi = 0$.

Again, for the circle map (1) steps 3–5 should be implemented. To calculate the dependence of the Lyapunov exponent on the control parameter these steps should be repeated for the several fixed values of the control parameter. Figure 2(a) illustrates the dependence of the Lyapunov exponent under consideration in the supercritical region ($\varepsilon > 0$) for the control parameter ε. The results of application of the proposed method are shown by filled circles and the solid line corresponds to the Lyapunov exponent computed by Eq. (4). We clearly see the good agreement between obtained data.

To illustrate the precision of the estimated value of the Lyapunov exponent under consideration the dependence of the relative error $\delta = |\hat{\Lambda}_0 - \Lambda_0|/|\hat{\Lambda}_0|$ (where Λ_0 is the estimated value of the Lyapunov exponent corresponding to the enslaved phase dynamics, and $\hat{\Lambda}_0$ is the Lyapunov exponent value obtained with the help of the standard algorithm) on the criticality parameter ε is shown in Fig. 2(b). As one can see, the relative error δ is small in the whole range of the criticality parameter values $\varepsilon > 0$ except for the narrow region in the vicinity of the bifurcation point ε_c where the value of δ tends to be large, since the proposed method becomes worse approaching the transition point as has been discussed above. In other words, the proposed method gives a good estimation of the considered Lyapunov exponent value in the large range of the control parameter values corresponding to the synchronous regime. In turn, the area in the vicinity of the critical point, where the relative error is sufficiently large, is small in comparison with the range of the control parameter values where the proposed method gives a good result. At the same time, in the subcritical regime the Lyapunov exponent under study is very close to zero, and, generally, there is no reason to estimate the Lyapunov exponent value there. As a consequence, the inapplicability of the proposed method in this region cannot be considered as a disadvantage.

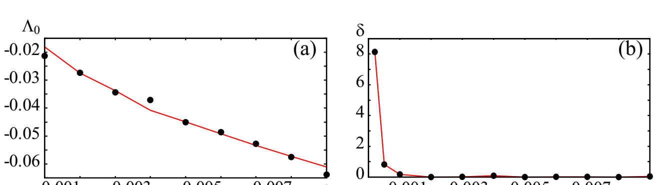

FIG. 2. (Color online) Dependencies of (a) Lyapunov exponent Λ_0 and (b) the relative error δ for the criticality parameter ε for the circle map (1) with $\Omega = 0.1$ in the presence of noise. In panel (a) data obtained by means of the proposed method are marked by filled circles; the results of application of a standard algorithm are shown by the solid line.

B. Van der Pole oscillator

As the next example we consider the classical Van der Pole generator

$$\ddot{x} - (\lambda - x^2)\dot{x} + x = \epsilon \sin(\omega t) + D\xi(t) \tag{10}$$

driven by the external harmonic signal in the presence of additive noise. Here $\lambda = 0.1$ is the control parameter determining the system dynamics, $\omega = 0.98$ and ϵ are frequency and amplitude of the external signal, respectively, $\xi(t)$ is δ-correlated Gaussian noise $[\langle\xi(t)\rangle = 0, \langle\xi(t)\xi(\tau)\rangle = \delta(t-\tau)]$, and D is its intensity. To integrate system (10) the Euler method with the time step $h = 5 \times 10^{-4}$ has been used.

With the increase of the external signal amplitude ϵ the initially zero Lyapunov exponent of system (10) passes in the field of the negative values. In the absence of noise ($D = 0$) it becomes negative for $\epsilon = \epsilon_c = 0.0238$ which corresponds to the synchronization onset in the system under study. The presence of noise results in the shift of the threshold value of the synchronous regime onset in the field of the bigger values of the external signal amplitude ($\epsilon_s = 0.029$ for $D = 1$). At that the transition of the Lyapunov exponent under study in the negative field takes place a little bit earlier in comparison with the noiseless case (see also Ref. [29]).

Let us apply the proposed method for estimation of the value of the conditional Lyapunov exponent corresponding to enslaved phase dynamics of system (10) with $D = 1$ from time realization for $\epsilon > \epsilon_s$. The principles of calculation of the Lyapunov exponent in such case are practically the same as in the case of the discrete maps considered above. The only one distinction consists in the fact that we should use the time dependence of the phase difference $\Delta\varphi(t)$ between the signal and the external force instead of the signal itself, and, therefore, all steps 1–5 must be done to estimate the value of the Lyapunov exponent corresponding to enslaved phase dynamics. Due to the fact that nonautonomous system demonstrates synchronous behavior for the selected values of the external signal amplitude the phase difference $\Delta\varphi(t) < 2\pi$ is locked, and the analyzed signal does not increase with time.

Figure 3(a) illustrates the time dependence of the phase difference $\Delta\varphi(t)$ of the nonautonomous Van der Pole generator for $\epsilon = 0.043$, whereas in Fig. 3(b) the probability distribution

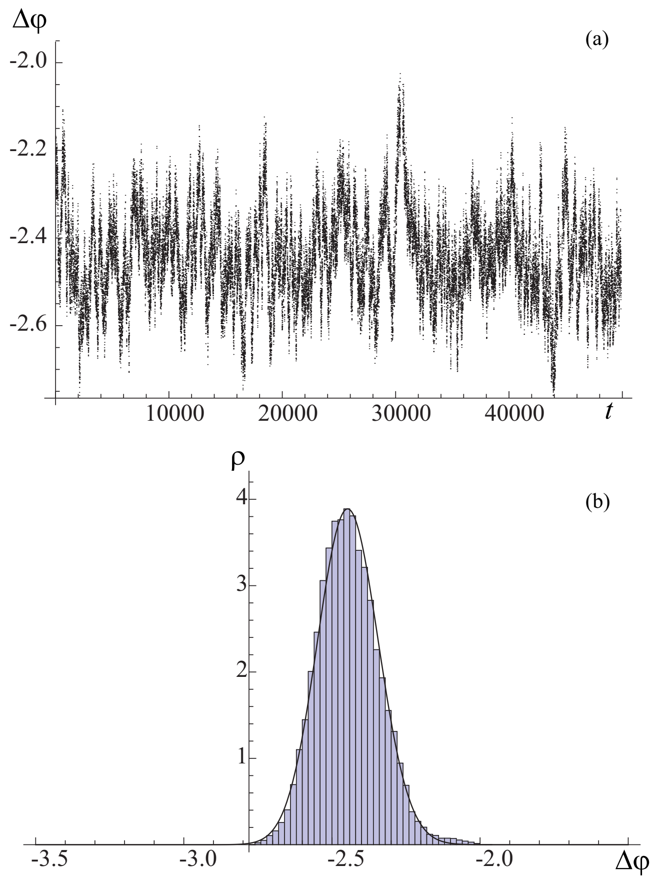

FIG. 3. (Color online) (a) Time dependence of the phase difference $\Delta\varphi(t)$ of the nonautonomous Van der Pole generator (10) for $D = 1$, $\epsilon = 0.043$; (b) probability density $\rho(\Delta\varphi)$ (histogram) obtained by the phase difference $\Delta\varphi(t)$ and its approximation by regularity (6) at $A = 1.47 \times 10^{-82}$, $\Omega = 0.004\,86$, $\varepsilon = 0.0029$, $D = 0.0005$ (solid line).

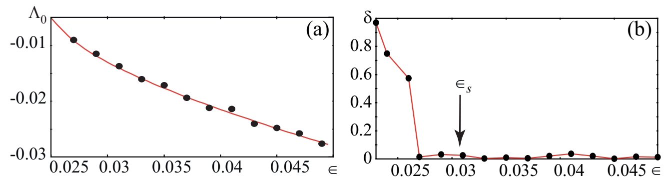

FIG. 4. (Color online) Dependencies of (a) conditional Lyapunov exponent Λ_0 corresponding to the enslaved phase dynamics of nonautonomous Van der Pole generator (10) in the presence of noise ($D = 1$) and (b) the relative error δ on the external signal amplitude ϵ. In panel (a) data obtained by means of the proposed method are marked by filled circles; the results of application of Benettin algorithm are shown by the solid line. The boundary of the synchronous regime in panel (b) is shown by the arrow.

of the phase difference $\Delta\varphi(t)$ for the same values of the control parameters and its approximation by the regularity (6) are shown. The search for the parameters of approximation has been performed in the same way as has been done for the quadratic and circle maps described above. Parameter $D = D_{\text{eff}} = 0.0005$ has been defined as an effective phase diffusion coefficient, whereas the other parameters $A = 1.47 \times 10^{-82}$, $\Omega = 0.004\,86$, and $\varepsilon = 0.029$ have been found by the approximation of the probability distribution density. Substitution of the values of such parameters in the relation (9) with $x_1 = -3$, $x_2 = -1$ gives the value of the conditional Lyapunov exponent $\Lambda_0 = -0.024$, which agrees well with the results of application of the Benettin algorithm [16].

Figure 4(a) shows the dependence of the considered conditional Lyapunov exponent of system (10) on the external signal amplitude ϵ in the supercritical region obtained by the proposed method (dots) and Benettin algorithm [16] (solid line). The relative error δ is given in Fig. 4(b). It is clearly seen that as in the case of the circle map a good agreement of the results of both methods takes place. As well as in the case of the discrete map considered above the relative error δ is small in the whole range of the criticality parameter $\epsilon > \epsilon_s$ values corresponding to the synchronous behavior of the driven Van der Pole oscillator in the presence of noise [the onset of the synchronous regime is shown in Fig. 4(b) with the arrow]. One can see that the value of δ tends to be large in the region corresponding to the pretransitional dynamics (so-called eyelet intermittency; see Refs. [45,55]). Comparing discrete map (1) with the dynamics of the driven noised Van der Pole oscillator (10) one also must take into account that the transitional point ε_c for (1) corresponds to the critical point ϵ_c for (10), but not the synchronization boundary ϵ_S, and, therefore, the proposed method is applicable in the whole region of the synchronous dynamics of the considered flow system.

C. Two coupled Rössler systems

As the last example we consider the results of application of the proposed method for the estimation of the value of the conditional Lyapunov exponent corresponding to the enslaved phase dynamics in the coupled chaotic systems. As the system under study we use two unidirectionally coupled Rössler systems

$$\begin{aligned} &\dot{x}_d = -\omega_d y_d - z_d, \quad \dot{y}_d = \omega_d x_d + a y_d, \\ &\dot{z}_d = p + z_d(x_d - c), \quad \dot{x}_r = -\omega_r y_r - z_r + \sigma(x_d - x_r), \\ &\dot{y}_r = \omega_r x_r + a y_r, \quad \dot{z}_r = p + z_r(x_r - c), \end{aligned} \tag{11}$$

where $a = 0.15$, $p = 0.2$, $c = 10.0$, $\omega_d = 0.93$, $\omega_r = 0.95$ are the control parameter values, and σ is a coupling parameter strength. The increase of the coupling parameter σ results in the transition of the conditional Lyapunov exponent under study in the field of the negative values. At that, as in the case of nonautonomous Van der Pole generator, such a transition precedes the phase synchronization regime onset in system (11). So one can estimate the value of the conditional Lyapunov exponent corresponding to the enslaved phase dynamics by the analysis of the phase difference between interacting systems in the same way as has been done for the Van der Pole generator considered above.

Figure 5(a) illustrates the time dependence of the phase difference $\Delta\varphi(t)$ of two coupled Rössler oscillators (11) for $\sigma = 0.07$, whereas in Fig. 5(b) the probability distribution of the phase difference $\Delta\varphi(t)$ for the same values of the control parameters and its approximation by regularity (6) are shown. The search of the parameters of approximation has

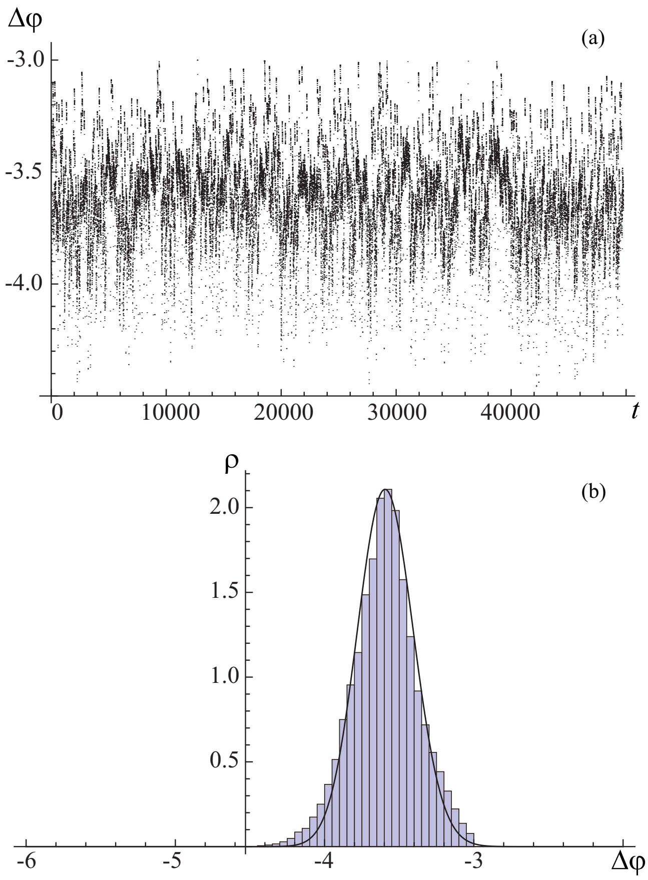

FIG. 5. (Color online) (a) Time dependence of the phase difference $\Delta\varphi(t)$ of Rössler oscillators (11) for $\sigma = 0.07$; (b) probability density $\rho(\Delta\varphi)$ (histogram) obtained by the phase difference $\Delta\varphi(t)$ and its approximation by regularity (6) at $A = 4.97 \times 10^{-54}$, $\Omega = 0.0048$, $\varepsilon = 0.0617$, $D = 0.0024$ (solid line).

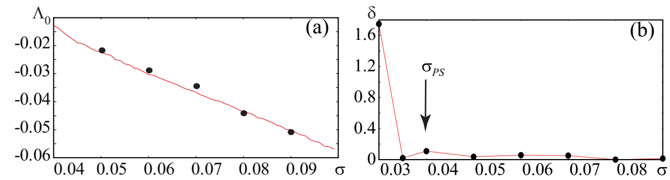

FIG. 6. (Color online) Dependencies of (a) conditional Lyapunov exponent Λ_0 corresponding to the enslaved phase dynamics of Rössler oscillators (11) and (b) the relative error δ on the coupling parameter σ. In panel (a) data obtained by means of the proposed method are marked by points, the results of application of Benettin algorithm with Gram-Schmidt orthogonalization procedure are shown by solid line. The boundary of the phase synchronization regime σ_{PS} in panel (b) is shown by the arrow.

been performed in the same way as has been done before for the systems described above. Again, as well as for the other systems considered above, there is a good agreement between the distributions $\rho(\Delta\varphi)$ obtained numerically and predicted theoretically.

It should be noted that the numerically obtained distribution contains small heavy tails that results in its negligible deviation from the theoretical regularity (6). At the same time, the presence of such deviations does not influence sufficiently the value of the Lyapunov exponent corresponding to the enslaved phase dynamics calculated with the help of the deduced distribution. In particular, Fig. 6(a) illustrates the dependence of the conditional Lyapunov exponent corresponding to the enslaved phase dynamics of Rössler oscillators (11) on the coupling parameter σ in the phase synchronization region obtained by the proposed method (marked by filled circles) and Bennetin algorithm with Gram-Schmidt orthogonalization procedure (solid line). The relative error value, δ, in turn, is given in Fig. 6(b). It is clearly seen a good agreement between obtained data in the whole range of the coupling parameter strength corresponding to the phase synchronization regime. As well as for the driven Van der Pole oscillator with noise (10) the relative error δ takes the large values in the region of the eyelet intermittency, whereas in the phase synchronization regime the relative error is close to zero and the estimated values of the Lyapunov exponent under study coincide with good accuracy with the corresponding values obtained with the help of Benettin algorithm and Gram-Schmidt orthogonalization procedure.

IV. CONCLUSION

In conclusion, the method for the estimation of the conditional Lyapunov exponent corresponding to the enslaved phase dynamics from time realization has been proposed. It allows to calculate the value of the Lyapunov exponent in the supercritical region of the control parameters in both the model systems with discrete time and nonautonomous noise-driven periodic and chaotic oscillators. Although having analyzed the error of the approximation of the probability distribution ρ by the analytical expression (6) one can localize the transition point, this localization is inaccurate and takes more time in comparison with the classical method of the phase difference examination [41]. As a consequence, the use of the proposed method for the transition point detection seems to be ineffective, whereas for the estimation of the Lyapunov exponent values for the synchronous regimes it gives the good results. As far as the subcritical regime (corresponding to the asynchronous dynamics) is concerned, the Lyapunov exponent under study is very close to zero there, and, therefore, the inapplicability of the proposed method below the transition point is not thought to be the disadvantage. In other words, the proposed technique is applicable for the broad class of the systems being in the regime of the phase synchronization. We believe the proposed method can also be applied to real experimental physical or physiological time series to define the degree of synchronism of the regime realized in the system.

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