AIP Chaos
An Interdisciplinary Journal of Nonlinear Science
25TH ANNIVERSARY

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Citation: Chaos **25**, 013118 (2015); doi: 10.1063/1.4907175
View online: http://dx.doi.org/10.1063/1.4907175
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/25/1?ver=pdfcov
Published by the AIP Publishing

CHAOS **25**, 013118 (2015)

Quantifying chaotic dynamics from integrate-and-fire processes

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(Received 29 November 2014; accepted 20 January 2015; published online 30 January 2015)

Characterizing chaotic dynamics from integrate-and-fire (IF) interspike intervals (ISIs) is relatively easy performed at high firing rates. When the firing rate is low, a correct estimation of Lyapunov exponents (LEs) describing dynamical features of complex oscillations reflected in the IF ISI sequences becomes more complicated. In this work we discuss peculiarities and limitations of quantifying chaotic dynamics from IF point processes. We consider main factors leading to underestimated LEs and demonstrate a way of improving numerical determining of LEs from IF ISI sequences. We show that estimations of the two largest LEs can be performed using around 400 mean periods of chaotic oscillations in the regime of phase-coherent chaos. Application to real data is discussed. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4907175]

Characterization of chaotic oscillations in complex nonlinear systems is easily performed when the equations describing the analyzing dynamical regime are known. Such a characterization is provided based on the Lyapunov spectrum that is clearly determined with the standard approach.[1] However, in many practical situations, as in Neuroscience or in Earth Sciences, the mathematical model is unknown. Instead often only a scalar time series representing the discretized phase space coordinate is available. Then, estimation of the Lyapunov exponents (LEs) can be provided for the reconstructed attractor. [2] A more complicated problem discussed in this paper is extracting complex dynamics from point processes such as, e.g., interspike intervals (ISIs). Here, we consider spiking events produced by the integrate-and-fire (IF) model. At a high firing rate, the reconstruction based on the measured output point process enables clear and correct estimation of the two largest Lyapunov exponents. However, quantifying complex oscillations has limitations at a low firing rate. The aim of this paper is to improve the quality of extracting chaotic oscillations from interspike intervals. We discuss features of reconstruction based on this type of point processes.

I. INTRODUCTION

Point processes in which information about systems dynamics is carried by times of some events are widely known in many areas of natural sciences.[3] A typical example is a sequence of electrical pulses of similar shape produced by a sensory neuron (a spike train). Such sequences encoding external stimuli are used by the central nervous system of a human or an animal as a source of information that provides an internal representation of the external world in the cortex of the brain.[4] A feature of this kind of data series is a limitation of available knowledge about the continuous-time evolution of the analyzed system. Understanding the processes of information encoding by neurons and their networks is still a challenging problem.

From a more general point of view, we may consider a threshold device with an input and an output: the input signal $S(t)$ reflects a continuous-time dynamics produced by a complex system, and the output point process T_i, $i=1, 2,\ldots,n$ is an available data series based on which we need to characterize dynamical features of $S(t)$. The works[5] addressed this problem in the context of the reconstruction of dynamical systems from ISIs $I_i = T_{i+1} - T_i$. Two types of quite simple models of spike generation were considered in these studies, namely, IF (or integrate-and-reset) and threshold-crossing (TC) models.[6] IF models describe a generation of spiking events when the integral from $S(t)$ reaches a given threshold level with the further resetting of the value of the integral. Besides spiking phenomena in neural networks, IF models are used, e.g., within delta-sigma data converters[7] and in many other applications. TC models assume a generation of spikes when a signal $S(t)$ crosses some threshold value Θ in one direction. If $S(t)$ is a low-dimensional chaotic signal, the obtained sequence of ISIs has a relation to return times into the Poincaré section.

Point processes generated by IF models represent a simpler case as compared with TC models. Thus, an embedding theorem was proved for IF ISIs[8] being an analogue to the Takens theorem for time series.[9] Therefore, an attractor reconstructed from an IF ISI sequence associated with chaotic oscillations $S(t)$ keeps metric and dynamical properties of the attractor related to the input dynamics. A reconstruction based on the sequences of I_i with the standard delay method provides a possibility to estimate metric characteristics of chaotic regimes such as, e.g., the correlation dimension.[5] A direct application of this method for determining dynamical properties (in particular, the Lyapunov spectrum) is less effective, and special approaches may be useful for increasing the quality of numerical estimations.[10,11] Thus, a transition from TC ISI sequences to time dependencies of the instantaneous frequency of oscillations provides an effective way to quantify chaotic and hyperchaotic regimes at the input of TC models.[12] For IF ISI sequences, a restoration of the input signal can be performed at high firing rates[12] and, therefore, the input dynamics can be quantified based on a variety of standard techniques proposed for time series.

Nevertheless, an ability of quantifying dynamical properties of a chaotic regime from IF ISI sequences at a low spiking rate is not obvious. In the previous studies,[11,12] limitations of the reconstruction techniques for point processes were discussed with main attention to TC ISIs. In particular, it was stated that the largest LE can be estimated from return times if the mean ISI does not exceed the prediction time for chaotic oscillations.[13] Moreover, such estimations are performed for both, chaotic and hyperchaotic regimes even if some oscillations are missed.[12] However, restrictions of extracting dynamics from IF ISIs were not studied in detail. In this work, we describe possibilities provided with the reconstruction methods based on IF ISI sequences. We discuss a way for increasing the precision of numerical estimations of LEs from IF point processes reflecting chaotic input dynamics and show that quite short data series are enough for a correct characterization of dynamical regimes of complex processes from IF ISIs.

This paper is organized as follows. In Sec. II, we describe a theoretical background of reconstruction based on IF ISI sequences. In Sec. III, we discuss peculiarities and limitations of extracting dynamics from point processes and consider how the obtained results depend on the algorithmic parameters. In Sec. IV, we briefly describe a possible application to natural systems. Some concluding remarks are given in Sec. V.

II. RESTORATION OF AN INPUT SIGNAL FROM INTEGRATE-AND-FIRE INTERSPIKE INTERVALS

Integrate-and-fire model of spike generation is widely used in neurobiology for describing neuron firings associated with voltage spikes.[4,6] It can be treated as a threshold system with an input signal $S(t)$ that is integrated from a time moment T_0. At the times T_i, $i = 1, 2,\ldots,n$, when the integral reaches a given value θ (a firing threshold), spikes are generated, and the integral is reset to zero (Fig. 1). This procedure is defined as

$$\int_{T_i}^{T_{i+1}} S(t)dt = \theta, \quad I_i = T_{i+1} - T_i. \tag{1}$$

Here, we consider a low-dimensional chaotic process as the input signal $S(t)$. Analysis of dynamical properties of $S(t)$ from an output sequence of timings T_i associated with spiking events is easily performed at high firing rates, i.e., for small interspike intervals I_i. In this case, the integral (1) can be estimated based on the rectangular rule

$$\int_{T_i}^{T_{i+1}} S(t)dt \simeq S\left(\frac{T_i + T_{i+1}}{2}\right)I_i. \tag{2}$$

Therefore, the signal $S(t)$ is restored from an IF ISI sequence (Fig. 2) as

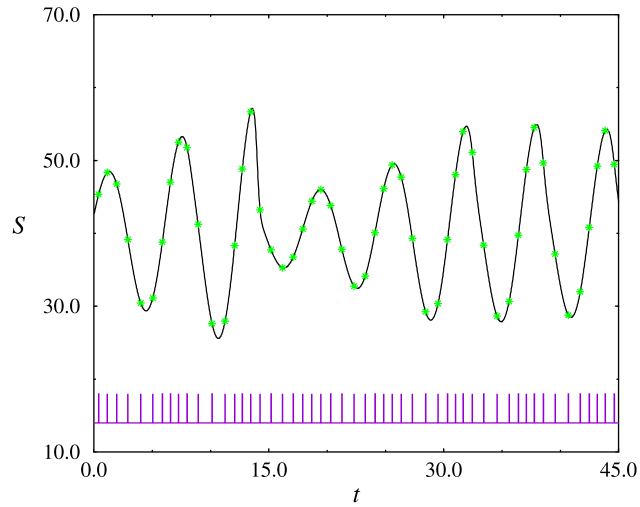

FIG. 1. Transformation of the input signal $S(t)$ into a sequence of spikes by the integrate-and-fire model (1).

$$S\left(\frac{T_{i+1} + T_i}{2}\right) \simeq \frac{\theta}{I_i}, \tag{3}$$

and the precision of such a restoration increases with reducing I_i.

An obvious limitation of the given approach occurs when the firing rate becomes small, and the error of the approximation (2) increases. According to the mean value theorem, time moments $\hat{t}_i$ can be introduced at which the values $S(\hat{t}_i)$ are estimated from ISIs

$$S(\hat{t}_i) = \frac{\theta}{I_i}, \quad T_i \le \hat{t}_i \le T_{i+1}. \tag{4}$$

Because we deal with point processes and information on the dynamics between the times T_i is not known, an uncertainty δ in determining of $\hat{t}_i$ appears

$$\hat{t}_i = \left(\frac{T_i + T_{i+1}}{2} + \delta_i\right). \tag{5}$$

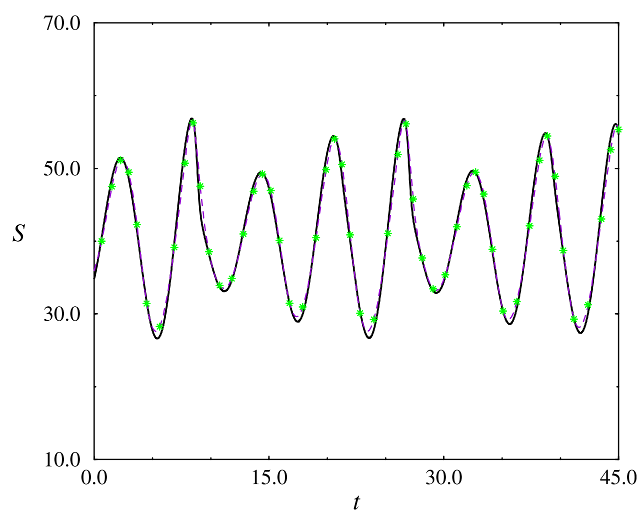

FIG. 2. Restoration of the input signal $S(t)$ from the output IF ISI sequence at a high firing rate. The original signal is shown by the solid line, and the signal restored from IF ISIs is given by the dashed line.

This uncertainty increases for larger I_i associated with the larger threshold θ (Fig. 3).

Larger δ imply larger errors in the restoration of sample points $S(\hat{t}_i)$. Further, we shall discuss limitations of the appropriate reconstruction of the attractor's dynamical characteristics based on IF ISI sequences at the increased threshold level. Since time intervals between the data points $S(\hat{t}_i)$ are varied, in order to use the standard approach for the reconstruction,[14] these samples are interpolated by a smooth function (e.g., by a cubic spline). The interpolation increases the number of points in the reconstructed phase space allowing a reduction of the orientation errors and provides a way to apply the widely used method for LEs estimation from time series.[2]

Note that the value of θ is unknown when dealing with the output point processes. Restoring the input signal, we can take $\theta = 1$ in Eq. (3). In this case, a linear transformation of the input signal $kS(t)$ is obtained; however, the value of $k = 1/\theta$ does not influence the further reconstruction and the estimation of LEs.

III. EXTRACTING DYNAMICAL CHARACTERISTICS FROM INTEGRATE- AND-FIRE POINT PROCESSES

Aiming to discuss abilities of quantifying dynamics based on IF ISI series, we consider the Rössler system as the source of chaotic oscillations at the input of the IF model

$$\frac{dx}{dt} = -(y+z), \quad \frac{dy}{dt} = x + ay, \quad \frac{dz}{dt} = b + z(x-c) \tag{6}$$

with the parameter set $a = 0.15$, $b = 0.2$, and $c = 10.0$ related to a chaotic regime. Avoiding negative values of the input signal, perform a translation of the coordinate $x(t)$ as $S(t) = x(t) + 40$. The threshold θ defines the firing rate of IF model. According to the description in Sec. II, an increased error of determining metric and dynamical characteristics is expected with growing θ.

In this study, we estimate Lyapunov exponents as numerical measures of complex dynamics produced by Eqs. (6) using the approach.[2] Within this approach, the largest LE is computed as an average rate of the exponential growth of small perturbations

$$r(t) = r_0 \exp\left[\lambda_1(t_0)(t - t_0)\right], \tag{7}$$

where r_0 represents the distance between the fiducial and a neighboring trajectories in the reconstructed phase space at the time moment t_0. Evolution of this perturbation is characterized by the increment $\lambda_1(t_0)$ that varies for different points in the phase space (the latter is indicated by its dependence on the time moment t_0). After averaging local values of $\lambda_1(t_0)$ along a typical phase trajectory, the largest LE is obtained. Note that the dependence (7) is valid only for small distances $r(t)$. If the value of $r(t)$ does not satisfy the condition of a linear approach (i.e., the exponential divergence of trajectories), renormalizations are required that assume selections of replacement vectors of smaller size. In general, a maximal available distance l between the trajectories can be introduced. If $r(t) > l$, renormalizations are performed.[2] Typically, l takes the value 5%–10% of the attractor size. Further, we show that this parameter is of a high importance when computing LEs from point processes. Besides, renormalizations are performed after a fixed time interval that provides a higher quality of estimations for inhomogeneous attractors. We used time intervals between renormalizations comparable with the mean period of oscillations.

At high firing rates, estimations of LEs from IF ISI series should be nearly close to the values computed from the coordinate $x(t)$ using the approach[2] or with the standard method.[1] Figure 4 verifies a correspondence between LEs estimated with the three mentioned techniques in a wide range of θ. Up to about $\theta = 60$, the error of determining LEs from IF point processes is quite small.

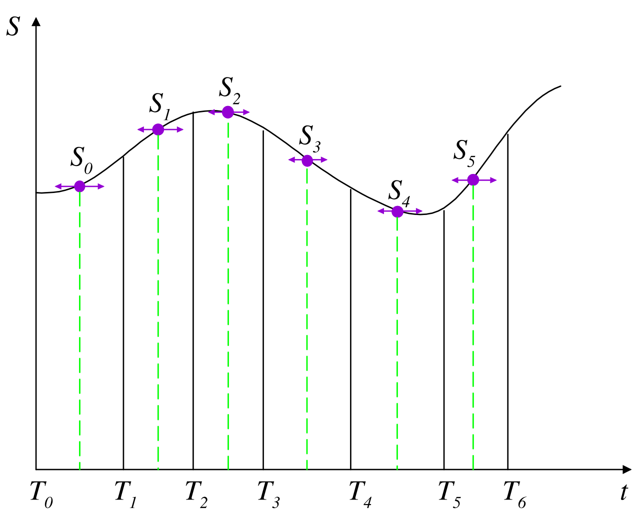

FIG. 3. Restoration of the input signal $S(t)$ from the output IF ISI sequence at a reduced firing rate. Arrows indicate uncertainties δ in positions of time moments $\hat{t}_i$ that increases with the threshold value θ.

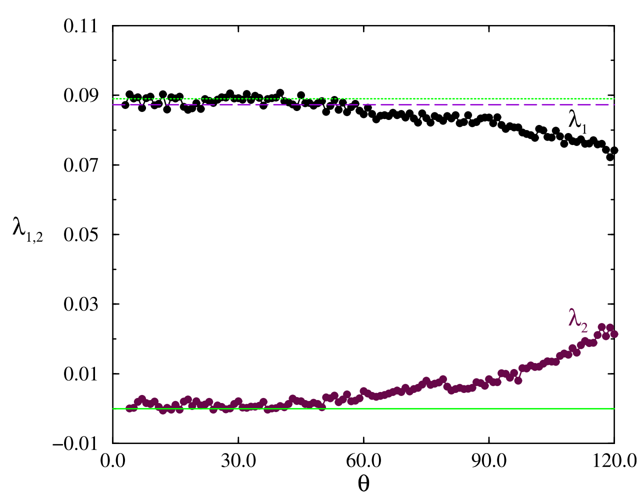

FIG. 4. Two largest LEs estimated from IF ISI series for different values of the threshold level θ. Dashed line indicates the expected value of $\lambda_1 = 0.0873$ obtained with the standard method.[1] Dotted line corresponds to the value $\lambda_1 = 0.0894$ computed from the coordinate $x(t)$ using the approach.[2]

The latter value of θ is associated with the mean IF ISI ($\bar{I}$) equal to about 25% of the averaged period of chaotic oscillations $x(t)$. Larger $\bar{I}$ does not allow a correct estimation of both, λ_1 and λ_2. According to Fig. 4, the second LE takes positive values in the region $\theta > 60$ providing spurious identification of the analyzed dynamical regime. Similar results are obtained for other sources of chaotic oscillations in the regime of a phase-coherent attractor. Thus, we consider the condition $\bar{I} < T_b/4$ as the limitation of an appropriate quantifying the chaotic dynamics from IF ISI sequences, where T_b is the mean period of the chaotic oscillations, i.e., the period associated with the basic frequency in the power spectrum.[12] Here, we discuss the case of weak (phase-coherent) chaos. For strongly developed chaotic dynamics, this condition may be corrected.

Independently of θ, LEs computed from IF ISI sequences do not demonstrate essential changes at the variation of the reconstruction parameters such as the time delay (τ) or the embedding dimension (d) (Fig. 5). Due to fluctuations of λ_1, averaging for different τ and d is necessary to obtain a reduced error of determining LE from the considered type of point processes. According to Fig. 5, underestimated values of λ_1 are obtained for large thresholds ($\theta > 60$), and an appropriate selection of the reconstruction parameters does not significantly improve the results.

Figure 6 illustrates how the value of λ_2 depends on the same reconstruction parameters. Note that in the region $\theta > 60$, the analyzed chaotic regime is wrongly diagnosed as hyperchaotic. Here, the selection of the time delay τ and the embedding dimension d do not essentially influence the obtained result (although λ_2 varies with τ and d, the value $\lambda_2 > 0$ is obtained for $\theta > 60$).

The selection of the algorithmic parameters becomes more important when considering the dependence of λ_1 on the maximal size of the perturbation vector l that defines the condition of a linear approach associated with the exponential growth of perturbations in the vicinity of the fiducial trajectory. According to Fig. 7(a), the value of l should appropriately be chosen for correct determining of λ_1.

Let us discuss features of the dependencies shown in Fig. 7(a) starting from the case $\theta = 5$ (marked by stars). For this threshold level, the firing rate is high (about 50 spikes per period of chaotic oscillations) and the uncertainty δ is fairly small and can be excluded from the consideration.

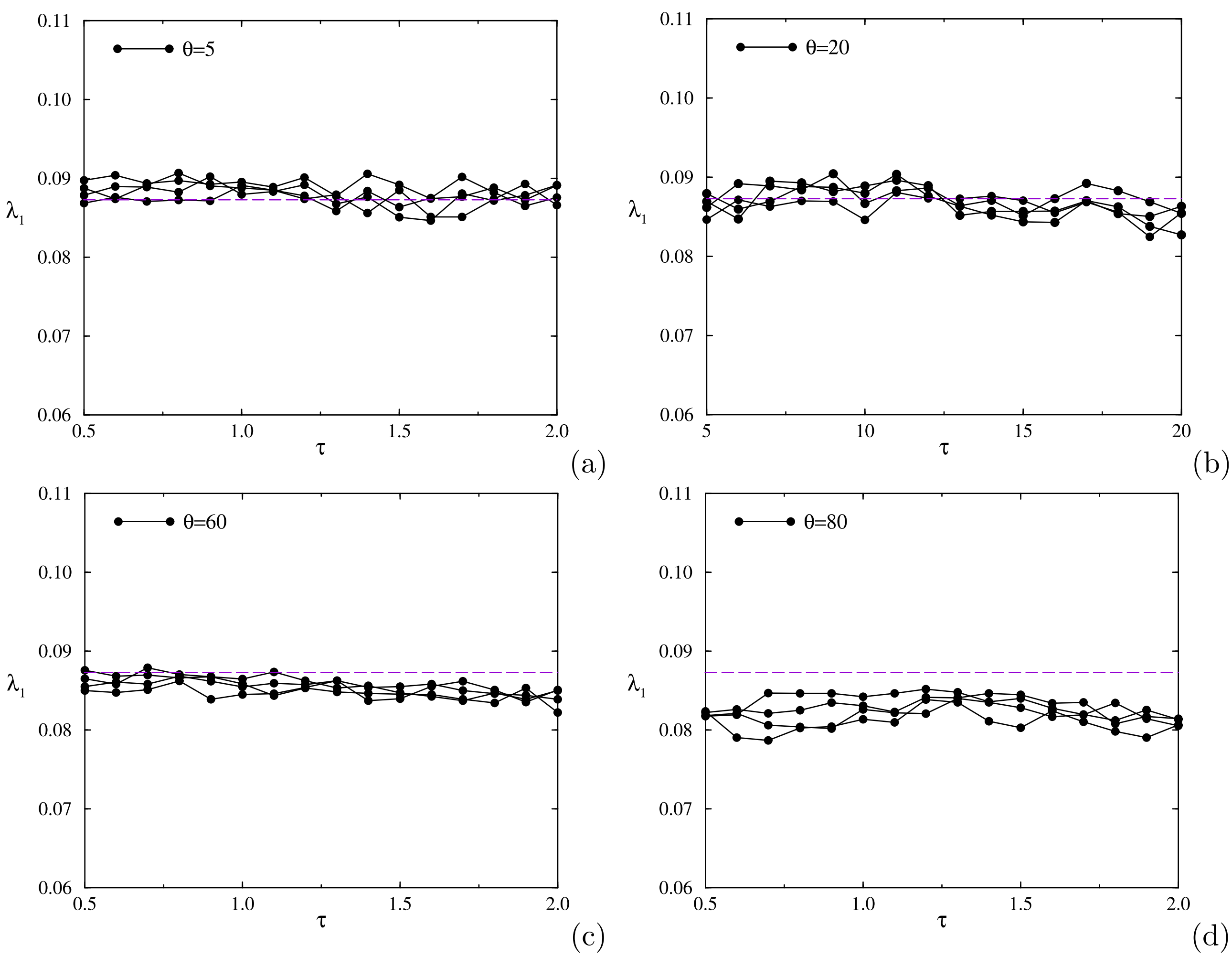

FIG. 5. The largest LE estimated from IF ISI series for different θ depending on the time delay τ between successive coordinates of the reconstructed vector. Circles show the values related to the embedding dimension $d \in [4, 8]$. The considered range of τ corresponds to about 8%–33% of the averaged period of chaotic oscillations. The values $\theta = 5$, 20, 60, 80 are related to the firing rate of about 50, 12, 4, and 3 spikes per mean period of oscillations, respectively. The dashed line is the same as in Fig. 4.

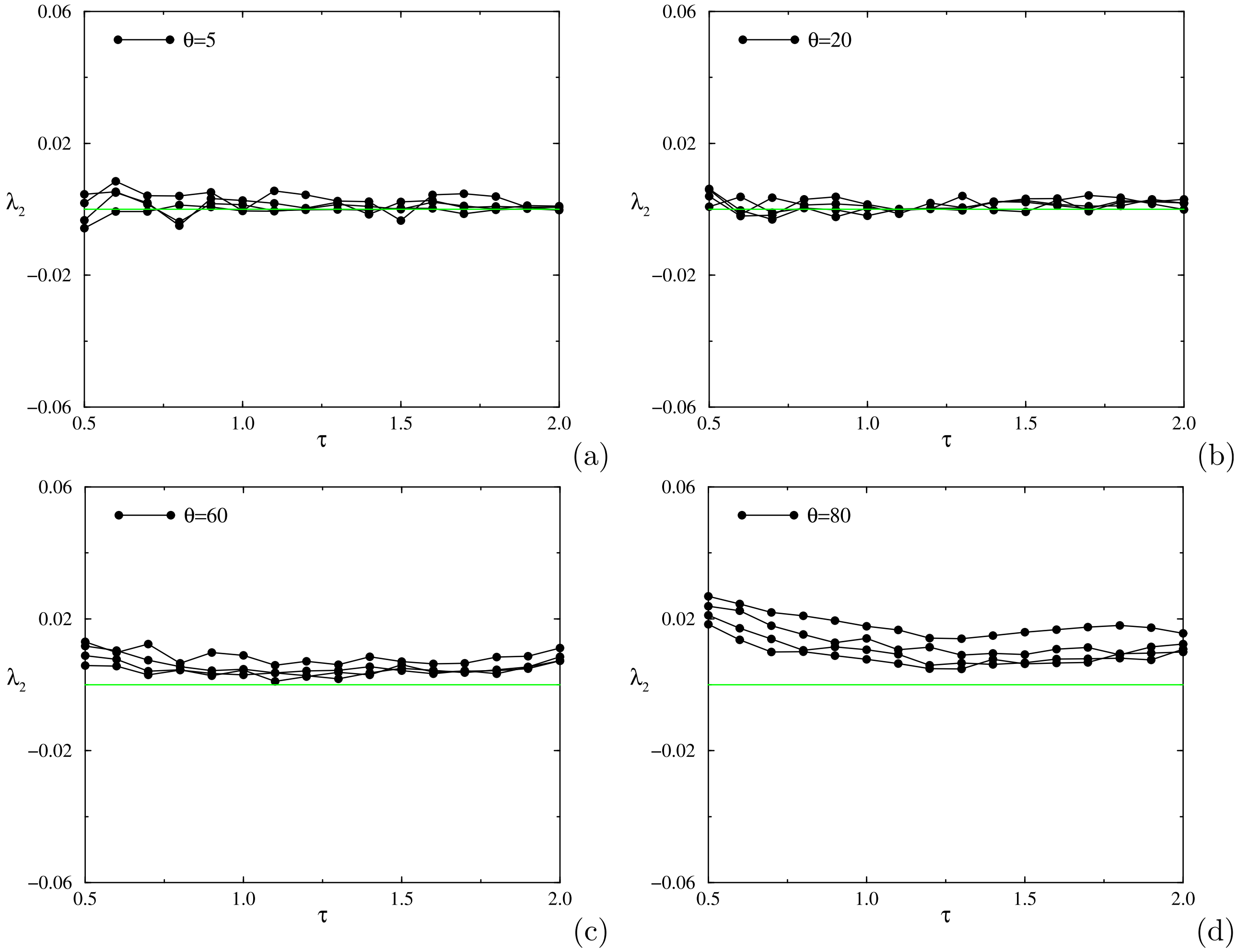

FIG. 6. The second LE estimated from IF ISI series for different θ depending on the time delay τ between successive coordinates of the reconstructed vector. Circles show the values related to the embedding dimension $d \in [4, 8]$. The considered range of τ corresponds to about 8–33% of the averaged period of chaotic oscillations. The solid line indicates zero value of the second LE.

There are two main factors reducing λ_1 in the regions of small and large l, respectively. For small l, underestimated values of λ_1 are caused by orientation errors occurring during replacements of the perturbation vectors in the reconstructed phase space.[2] The less l is, the more often are the replacement procedures performed, and the corresponding error can be accumulated in the course of averaging of the rate of trajectories divergences.

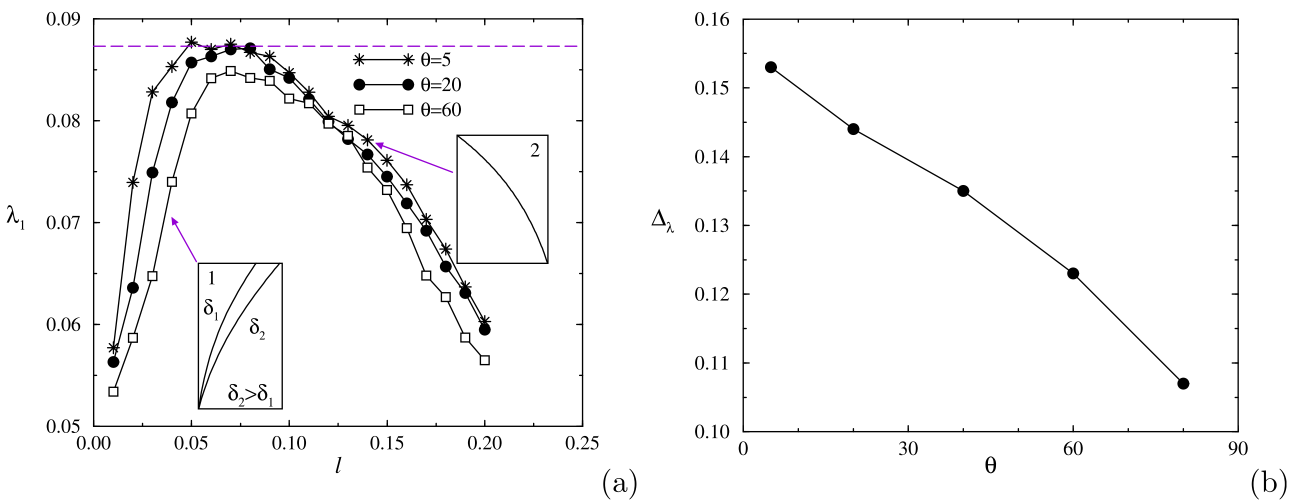

FIG. 7. The largest LE estimated from IF ISI series depending on the maximal size of the perturbation vector l (a), and the width of the corresponding dependence for different threshold levels (b). The value of Δ_λ is estimated at the level 80% of the maximum of $\lambda_1(l)$. The dashed line is the same as in Fig. 4.

For large l, the value of λ_1 is restricted by the condition of a linear approach. When the distance between the fiducial and a neighboring trajectory increases up to about 10% of the attractor size, the divergence of trajectories becomes not exponential leading to underestimated values of λ_1. In this case, the vector size before the replacement is typically less than the expected value. Such limitation may be roughly described by the dependence

$$\lambda_1(l) \sim \frac{1}{t^*}\ln(A - B * l), \tag{8}$$

where t^* is the time between replacements, A and B are some constant values, $A \gg B$. The dependence (8) is illustrated by the inset 2 in Fig. 7(a). The latter limitation always occurs when computing LEs from time series with the approach.[2] The interplay between two considered factors reduces λ_1, and the estimated value may become less than the expected LE.

When dealing with IF ISI sequences considered at large θ, an additional factor appears that limits the value of λ_1 in the range of small l. In this region, the size of the perturbation vector r_0 becomes comparable with the value of δ characterizing uncertainty in determining the time moment associated with the current value of $S(t)$ (Fig. 3). Within the first approximation, when considering equal uncertainty for the replacement vector and the perturbation vector before the replacement, λ_1 can be roughly estimated as follows

$$\lambda_1(l) \sim \frac{1}{t^*}\ln\left(\frac{l+\delta}{r_0+\delta}\right). \tag{9}$$

The latter dependence reduces λ_1, and the resulting LEs for larger δ associated with larger θ decrease what is illustrated by the inset 1 in Fig. 7(a). This inset shows dependencies of scaling coefficients that reduce λ_1 for different l and two values of δ.

Aiming to characterize the dependence $\lambda_1(l)$ (Fig. 7(a)), its width Δ_λ is introduced. Here, we consider Δ_λ as the distance between two values of l related to the level 80% from the maximum of $\lambda_1(l)$, i.e., the values $\lambda_1 \simeq 0.07$. Let us note that the width Δ_λ of the dependence $\lambda_1(l)$ decreases with the growing threshold level (Fig. 7(b)) providing a way for quantifying effects of uncertainties δ leading to underestimated LEs. Thus, computing of the dependencies $\lambda_1(l)$ allows estimating more precise values of LEs (related to their maxima) and characterizing effects of low spiking rate by reducing Δ_λ.

Aiming to avoid possible effects of short data series, estimations of LEs in Figures 4–7 are performed for sequences consisting of 10 000 IF ISIs. From numerical estimations, we have found, however, that a reduced number of data points is enough for a correct determining of Lyapunov exponents from integrate-and-fire point processes. Figure 8 demonstrates the dependencies of λ_1 and λ_2 versus the duration n of IF ISI sequences.

Both LEs are close to the expected values marked by dashed lines at large n. A good precision of determining λ_1 (with an error less than 10%) is obtained for $n > 1500$ IF ISIs

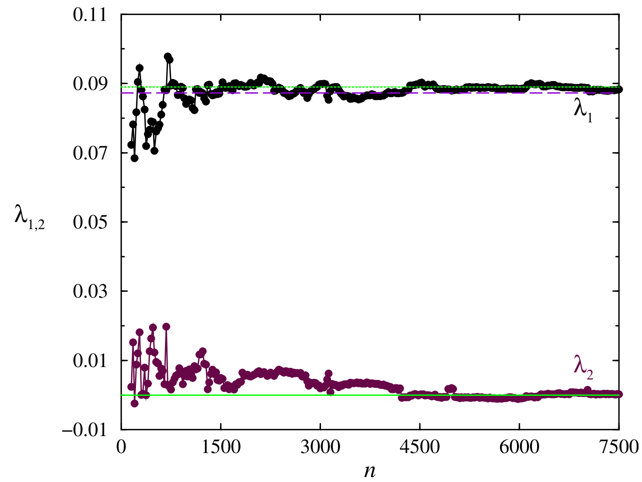

FIG. 8. Two largest LEs estimated from IF ISI series depending on the number of data points n for $\theta = 20$. The used definitions are the same as in Fig. 4. Here, we considered about $3 * n$ data points after the interpolation. The results are nearly insensitive to the selected interpolation step as compared with the maximal size of the perturbation vector.

that corresponds to about 125 mean periods of chaotic oscillations (T_b). If the two largest LEs need to be estimated, e.g., to clearly separate between chaotic and hyperchaotic dynamics, the length of data series should be increased up to about 4500 IF ISIs.

IV. APPLICATION TO REAL DATA

The considered approach can be used to analyze real data; however, its several features need to be mentioned. First of all, we should be careful with interpretations of the obtained results. Thus, for natural systems, we cannot be sure in the exponential divergence of trajectories in the reconstructed time space (due to noise, nonstationarity, etc.). That is why the values of $\lambda_{1,2}$ are better to interpret as numerical measures quantifying complexity (or instability) of the analyzed regime. Second, the threshold value is an unknown quantity when dealing with the output point processes. However, the knowledge of the threshold value is not important for the estimation of $\lambda_{1,2}$ from ISI sequences and the latter can be computed if the firing rate is quite high.

As an example of real data, let us consider an electrocardiogram (ECG) and perform estimations of $\lambda_{1,2}$ from the whole ECG (Fig. 9) and from a sequence of beat-to-beat intervals being times between the consequent R-peaks. For this purpose, we used ECGs of 5 young (20–22 years) healthy humans recorded under normal conditions. In the case of the whole ECG-recordings, we obtained the values (indicated as mean ± SE) $\lambda_1 = 0.49 \pm 0.12$ and $\lambda_2 = 0.28 \pm 0.09$. Applying the considered approach for about 1000 beat-to-beat intervals representing an example of the output point process, we got quite similar values $\lambda_1 = 0.46 \pm 0.14$ and $\lambda_2 = 0.23 \pm 0.12$. Thus, the consideration of point processes provides similar quantities of complexity as the analysis of the whole ECG. These quantities can be used to characterize the state of an organism under different physiological conditions.

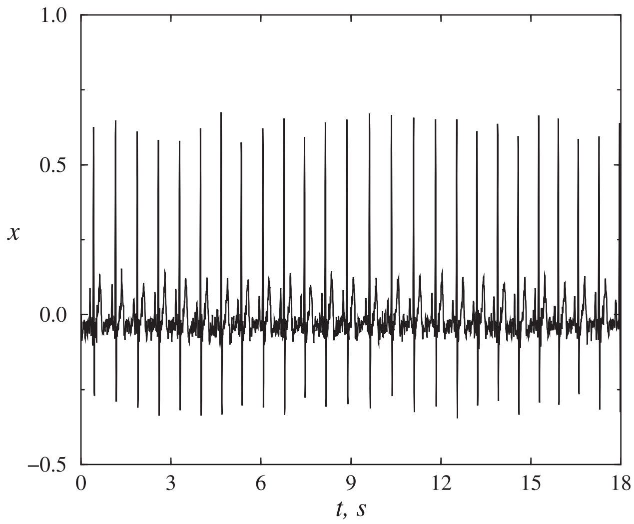

FIG. 9. An example of the considered ECG recording (a short fragment). The value of x is measures in arbitrary units.

V. CONCLUSION

In this study, we have considered potentials and limits of extracting dynamical features of chaotic processes at the input of the IF model from the output sequences of interspike intervals. Although this problem can easily be solved at high firing rates, its solution becomes more complicated when the spiking activity reduces. As a result, the diagnosed chaotic regime at the input of IF model can wrongly be characterized as hyperchaotic if the mean ISI exceeds the value of about $T_b/4$, where T_b is the mean period of oscillations in the regime of phase-coherent chaos.

We have described features of the dependence of λ_1 on the maximal distance between the trajectories in the reconstructed phase space l associated with the limits of the linear approach, and we have shown that the width of this dependence decreases with growing θ. On the one hand, this dependence provides an opportunity of selection the optimal parameter l leading to a more precise estimations of LEs. Thus, taking into account restrictions occurring for small and large l, the better selection of l is associated with the maximum of $\lambda_1(l)$. On the other hand, a reduced width of the given dependence characterizes effects of uncertainties occurring at a low firing rate. Low values of Δ_λ may serve as a sign of underestimated LEs in the course of their computing with the standard method.[2]

Unlike the case when the equations describing the analyzed dynamical regime are known, estimations of LEs from time series with the reconstruction technique are accompanied by orientation errors when performing renormalizations of perturbations in the reconstructed phase space. These errors have a tendency to essentially accumulate for each sequential LE. Due to this, we have restricted by only two LEs that can be estimated with an appropriate quality.

Based on the obtained results, we can also conclude that a quite short the input signal is enough for a correct characterization of the analyzed dynamical regime. Thus, if only the largest LE should be estimated from IF point processes, this can be done with an error of 10% from an input signal consisting of about 120–150 mean periods of oscillations. In order to clearly distinguish between chaotic and hyperchaotic regimes from IF ISI sequences, the duration of the input signal should be increased up to about 350–400 mean periods. This approach can be used as an alternative to other recent methods for characterizing chaos-hyperchaos transitions.[15] It can be applied to real data aiming to characterize, e.g., the state of a natural system under different conditions.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation (agreement No. 14-12-00224).

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