ISSN 1063-7850, Technical Physics Letters, 2017, Vol. 43, No. 1, pp. 107–109. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © Ya.Kh. Mohammad, O.N. Pavlova, A.N. Pavlov, 2017, published in Pis'ma v Zhurnal Tekhnicheskoi Fiziki, 2017, Vol. 43, No. 2, pp. 45–51.

Estimating the Predictability Time of Noisy Chaotic Dynamics from Point Sequences

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Received March 29, 2016

Abstract—A method for increasing the accuracy of estimation of the predictability time of noisy chaotic dynamics from system-related point sequences is proposed. General laws observed in the application of this method to interspike interval series of model threshold devices of two types operating in the regime of phase-coherent chaos are illustrated.

DOI: 10.1134/S1063785017010242

The main peculiarity of dynamical systems exhibiting chaotic regimes consists in that small changes in the initial conditions and/or parameters of the system lead to loss of predictability of the oscillatory process in some time. The degree of predictability will also change if the chaotic regime is studied in the presence of additional sources of fluctuations. To estimate the maximum time for which the system behavior can be predicted, a concept of predictability-time horizon is introduced [1] that depends on a priori data concerning the system studied and the statistics of noise. If the prognosis is based on experimental data, the predictability time τ_p is frequently defined as the "Lyapunov time"—a time interval for which the distance between adjacent phase trajectories increases by a factor of *e*. Estimation of the predictability horizon for a system with chaotic dynamics as the inverse of the maximum Lyapunov exponent ($\tau_p = 1/\lambda_1$) implies that an error in determining the predictability time will arise provided that calculation of the maximum exponent gives an incorrect λ_1 value.

The accuracy of calculation of the λ_1 value depends on the kind of information available about the system dynamics. If the equations of a mathematical model are known, the spectrum of Lyapunov exponents can be calculated with required accuracy using a standard algorithm [2]. When an analysis is carried out using time series of a dynamic variable, then methods of dynamical system reconstruction have to be employed [3–5]. The problem of calculating Lyapunov exponents becomes more complicated if the dynamic variable is subject to transformations, leading to decrease in the volume of information available on the system dynamics. In particular, chaotic signal transformation by threshold devices [6–9] results in that information on the system dynamics acquires the form of time series of stereotype pulses (interspike intervals, ISIs) generated upon crossing the threshold, which are referred to below as "point sequences."

The possibility of using point sequences to estimate the characteristics of chaotic dynamics, such as the correlation dimension and Lyapunov exponents, has been considered previously [6–13], and the conditions have been determined under which the dynamic regime at the input of a threshold device can be correctly identified. At high frequencies of pulse generation by a threshold device, the Sauer theorem is valid that is applicable to point sequences of the "integrate-and-fire" (IF) model type [10]. In the case of relatively low frequencies and other models of threshold devices, e.g., of the "threshold crossing" (TC) type, the possibility of calculating Lyapunov exponents has been confirmed by numerical simulations [11–15]. However, the aforementioned works did not take into account the presence of measurement noise in the input oscillatory process, which leads to additional fluctuations of the ISI duration.

The present investigation was intended to modify the method of determining the maximum Lyapunov exponent [5] so as increase the accuracy of estimation of the time of chaotic-dynamics predictability in the case of a noisy-point process and the absence of data on noise intensity.

The main idea of the proposed approach is as follows. The standard algorithm [5] stipulates the calculation of λ_1 as the average rate of exponential expansion of trajectories in a reconstructed phase space. During the analysis of one-dimensional projections of phase trajectories belonging to the chaotic attractor upon reconstruction, the boundaries of linear approx-

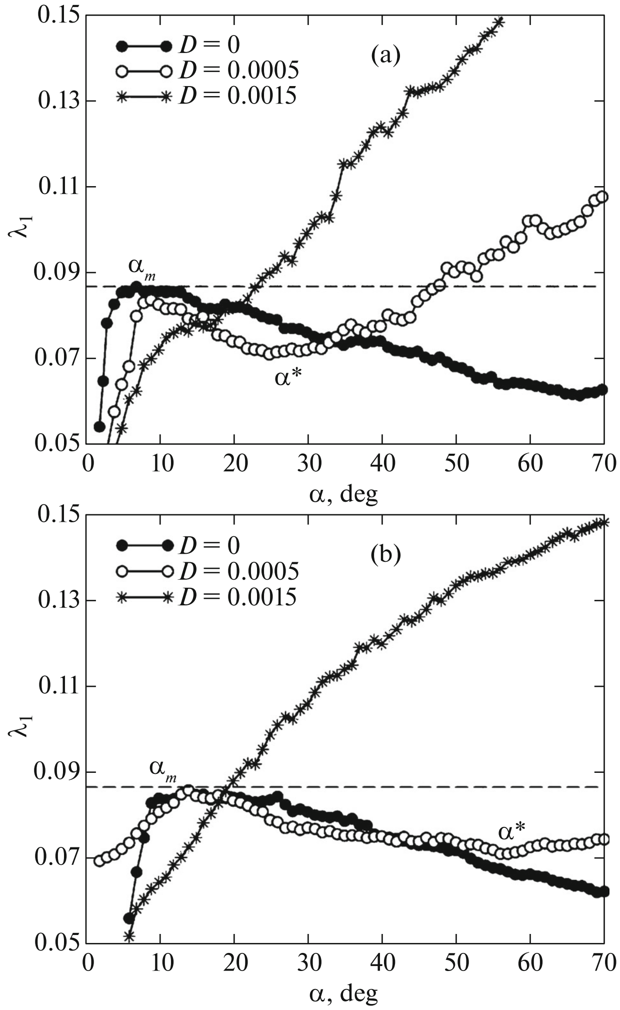

Fig. 1. Plots of the maximum Lyapunov exponent λ_1 vs. maximum error α of perturbation-vector orientation for chaotic oscillations in Rössler system (1) calculated from ISI sequences of (a) IF- and (b) TC-type models.

imation $[l_{\min}, l_{\max}]$ for perturbation vector $r(t)$ are determined that correspond to the expansion of trajectories related to the system dynamics. Above the upper boundary ($l_{\max}$), the estimate of λ_1 is underestimated because of a nonlinear limitation of the perturbation-vector size. The lower boundary ($l_{\min}$) is introduced in order to eliminate the additional expansion of trajectories caused by the presence of measurement noise in the signal under consideration. The algorithm [5] is also applicable to the analysis of point sequences upon their preliminary processing. In case of the IF model describing the generation of pulses at time moments T_i, when the integral of input signal S reaches threshold level θ, the input signal is reconstructed as $S(T_i) = \theta/(T_{i+1} - T_i)$ with an accuracy that grows with increasing generation frequency [7]. For a TC-type model in which pulses are generated when the input signal crosses the threshold level, the processing consists in approximation of the average instantaneous frequency as $\omega(T_i) = 2\pi/(T_{i+1} - T_i)$ [11]. Then, we pass to a signal with uniform sampling, which is determined by interpolation of $S(i\Delta t)$ and $\omega(i\Delta t)$ for IF and TC models, respectively, and finally analyze the results using the standard method [5].

The presence of input noise leads to some difficulties in calculations. In order to increase the accuracy of calculation of the maximum Lyapunov exponent, it is proposed to analyze the dependence of λ_1 in maximum error of orientation α determined as the angle between perturbation vectors before and after renormalization. From general considerations, it can be suggested that very small and very large α values would lead to a decrease in λ_1. Therefore, the Lyapunov exponents should be calculated for intermediate orientation angles.

Figure 1a shows the results of calculation of the maximum Lyapunov exponent for a phase-coherent chaos in the Rössler model

$$\frac{dx}{dt} = -y - z, \ \frac{dy}{dt} = x + ay, \ \frac{dz}{dt} = b + z(x - c), \quad a = 0.15, \ b = 0.2, \ c = 10. \tag{1}$$

Calculations were performed for a sequence of 2000 ISI values at the output of an IF-type model with noisy input signal $S(t) = x(t) + C + D\xi(t)$, where C is a constant coefficient introduced so as to avoid negative values of the input signal (in these calculations, C = 35) and $\xi(t)$ is the white noise. The results were obtained for $l_{\min} = 0.01$ and $l_{\max} = 0.1$. For the sake of convenience, calculations were performed upon signal $S(i\Delta t)$ normalization to a unit interval.

The initial $\lambda_1(\alpha)$ curve refers to dynamics in the absence of noise ($D = 0$). In this case, maximum $\lambda_1(\alpha)$ is observed at $\alpha = \alpha_m$ corresponding to the theoretically predicted maximum exponent calculated using the mathematical model (1) by the algorithm [2]. In the given example, $\lambda_1 = 0.087$ is indicated by the dashed line in Fig. 1a. On the left from this maximum, λ_1 is underestimated because of a low probability of selecting small values of the perturbation vector and high probability of going outside the linear approximation. On the right from this maximum, increasing error of the orientation vector makes it also necessary to take into account the expansion of trajectories in directions perpendicular to that of maximum expansion.

It is important to note that the initial $\lambda_1(\alpha)$ curve has a characteristic form that does not qualitatively change for systems with different chaotic behavior (in addition, we have studied chaotic oscillation regimes in the Lorenz model, oscillator with inertial nonlinearity, and Rössler model with weak and developed chaos). Estimation of the predictability time according to Fig. 1a yields $\lambda_p(\alpha_m) = 1/\lambda_1(\alpha_m)$. A deviation from α_m implies that τ_p given by this formula will significantly exceed the duration of prognosis calculated using model system equations (1) and lead to incorrect conclusions concerning the degree of determinacy of the analyzed dynamics.

The character of the $\lambda_1(\alpha)$ dependence changes in the presence of noise in the ISI sequence, which may be related to interferences in the input signal and/or fluctuations of the threshold level. Upon attaining the α^* value, the $\lambda_1(\alpha)$ curve exhibits positive slope for $\alpha > \alpha^*$. The α^* value tends to α_m with increasing noise intensity. This behavior of $\lambda_1(\alpha)$ is also characteristic of all the aforementioned examples of systems with chaotic dynamics and the presence of noise in the ISI sequence. Based on these laws, it is possible to use the $\lambda_1(\alpha)$ curve shape to draw qualitative conclusions about the presence of noise and its intensity, since increasing D leads not only to a decrease in α^*, but also to growth in the slope of $\lambda_1(\alpha)$ at $\alpha > \alpha^*$. Beginning with certain noise intensity D, the maximum of $\lambda_1(\alpha)$ curve disappears and the estimates of predictability time cease to be reliable. For the example in Fig. 1a, the results fail to be correct at $D = 0.0015$, which corresponds to the ratio of noise intensity to average ISI duration of 5×10^{-4}.

Analogous laws have been also revealed in the case of a TC-type model. Figure 1b shows an example of calculations of the maximum Lyapunov exponent for a phase-coherent chaos in system (1) in the absence of noise and in the presence of threshold fluctuations in the ISI sequence. The initial signal represented $x(t)$ variable and the threshold was set at $\theta = 0$. In the absence of noise, this system also exhibits a characteristic maximum of $\lambda_1(\alpha)$ (minimum of $\tau_p(\alpha)$), which corresponds to theoretically predicted estimates (dashed line). The presence of noise leads to changes in the slope for $\alpha > \alpha^*$, and this slope (as well as the α^* value) varies with increasing noise intensity.

Thus, we have described a modified method for calculation of the maximum Lyapunov exponent, which is based on the construction of a plot of λ_1 versus error α of the orientation of perturbation vector in the reconstructed phase space. This approach revealed the characteristic behavior of λ_1, which that can be used for drawing conclusions on the presence of noise in the system. If data for various levels of noise are available, the results can be qualitatively compared in terms of their intensity. By selecting parameter α corresponding to the maximum of $\lambda_1(\alpha)$ curve, it is possible to increase the accuracy of estimating the predictability time for the analysis of various types of noisy-point sequences.

Acknowledgments. This study was supported in part by the Russian Science Foundation, project no. 14-12-00224.

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Translated by P. Pozdeev