ISSN 1063-7850, Technical Physics Letters, 2015, Vol. 41, No. 11, pp. 1058–1060. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © A.N. Pavlov, O.N. Pavlova, Ya.Kh. Mohammad, 2015, published in Pis'ma v Zhurnal Tekhnicheskoi Fiziki, 2015, Vol. 41, No. 21, pp. 74–79.

Errors of Analysis of Parameters of Complex Oscillation Regimes Using Point Sequences of the Integrate-and-Fire Model

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Received May 27, 2015

Abstract—The problem of calculation of dynamical parameters of chaotic regimes of self-sustained oscillations using point processes is discussed. The "integrate-and-fire" model is used to exemplify the constraints of the method for attractor reconstruction using a sequence of time intervals between the time instants of pulse generation. The conditions of validity for calculation of the largest Lyapunov exponent and recommendations for the most accurate determination of dynamical parameters for complex oscillatory regimes in dynamical systems reconstruction using point processes are formulated.

DOI: 10.1134/S1063785015110073

Point processes in which information on the system dynamics is contained in time instants of occurrence of some events (for example, single pulse generation instants) are widely known in many fields of science and technology [1]. One of the conventional models of point pulse generation is the integrate-and-fire model, which is capable of describing the dynamics of systems of different nature: from delta–sigma converters applied in communication engineering [2] to the electric activity of neurons and their ensembles [3–5]. The integrate-and-fire model implies integration of signal $S(t)$ at the input of the threshold device,

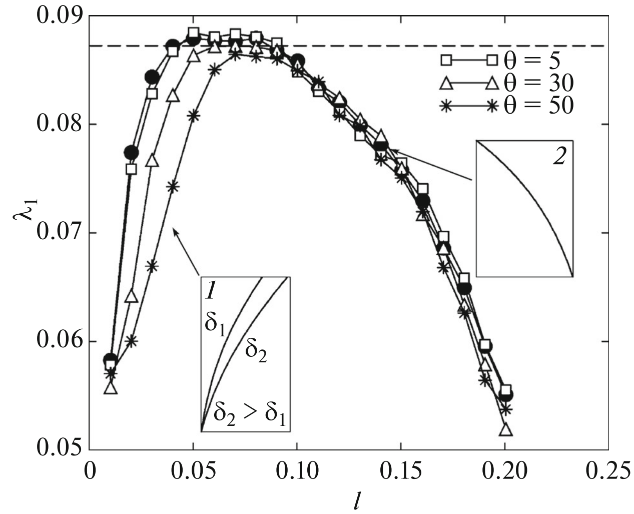
$$\int_{T_i}^{T_{i+1}} S(t)dt = \theta, \tag{1}$$

beginning from time instant T_i. When integral (1) reaches fixed threshold value θ, a single pulse is generated and the integral value is set to zero. The time intervals between the pulses $I_i = T_{i+1} - T_i$ are the carriers of information on the input process, and, with this information available, it is necessary to quantitatively describe the dynamics of the process $S(t)$ at the input of the threshold device. In the framework of the dynamical systems theory, the problem of reconstructing the attractor corresponding to the dynamical regime $S(t)$ can be solved based on intervals I_i [6–8], and its metric and dynamical characteristics can be calculated [8–10], including the correlation dimension, Lyapunov exponents, etc. Strict theoretical results substantiating the fundamental possibility of reconstruction were obtained in the framework of Sauer's theorem [11], which is a generalization of Takens' theorem [12] to the case of point processes. This theorem, however, holds only under the condition of high pulse generation rate in the integrate-and-fire model. In the case of low generation rate the possibility of attractor reconstruction can only be verified numerically. Numerical studies performed for the integrate-and-fire and other models of threshold systems established the possibility of diagnostics of chaotic and hyperchaotic dynamics regimes for point processes [13–17]. The domain of their applicability for the integrate-and-fire model, however, has not been studied in detail.

In this letter, we discuss the capabilities and constraints of quantitative description of dynamical parameters of complex oscillation regimes for point processes of the integrate-and-fire model. For a high pulse generation rate, integral (1) can be approximately calculated based on a simple variant of numerical integration, the method of rectangles,

$$\int_{T_i}^{T_{i+1}} S(t)dt \cong S\left(\frac{T_i + T_{i+1}}{2}\right) I_i \Rightarrow S\left(\frac{T_i + T_{i+1}}{2}\right) \cong \frac{\theta}{I_i}. \tag{2}$$

If the pulse generation rate is high, i.e., I_i takes small values (conditions of applicability of Sauer's theorem [11]), the accuracy of attractor reconstruction for the point process is high. With decreasing generation rate, approximate equality (2) does not hold any more. In accordance with the mean value theorem, there exist time instants $T_i \leqslant \hat{t}_i \leqslant T_{i+1}$ for which the values of the input process $S(\hat{t}_i)$ can be exactly reconstructed, $S(\hat{t}_i) = \theta/I_i$. However, since information on the dynamics between instants T_i is absent in the analysis



The largest Lyapunov exponent as a function of the parameter defining the domain of applicability of linear approximation for the pulse generation rates corresponding to the threshold values $\theta = 5$, 30, and 50. Black circles show the results of calculation using the signal $x(t)$ of model (3). Constraints (5) and (4) are shown in insets *1* and *2*. Dashed line shows the factor calculated using the equations of model (3).

of point processes, there appears uncertainty δ in determining corresponding times $\hat{t} = (T_i + T_{i+1})/2 + \delta_i$. Let us consider the consequence of this uncertainty in calculation of the standard parameter of chaotic oscillation regimes, the largest Lyapunov exponent. Let us take the following input signal as an example: $S(t) = x(t)$, where $x(t)$ is the coordinate of Rössler model,

$$\frac{dx}{dt} = -(y + z), \quad \frac{dy}{dt} = x + ay, \qquad \frac{dz}{dt} = b + z(x - c), \tag{3}$$

for $a = 0.15$, $b = 0.2$, and $c = 10$. Threshold level θ determines the pulse generation rate in the integrate-and-fire model. The studies demonstrated that, in the range $\theta < 0.53$, which corresponds to approximately four pulses per characteristic oscillation period, the application of the reconstruction method [10] provides correct estimation of the largest Lyapunov exponent (λ_1). This can be performed, however, only for appropriately chosen parameters of the algorithm for calculating the largest exponent [18], first of all, the domain of applicability of linear approximation (l) in determining the average exponential rate of trajectories divergence (see figure). Our studies explain such a dependence of the results on parameter l.

Let us first consider the case $\theta = 5$, which corresponds to generation of approximately 43 pulses per characteristic period of chaotic oscillations. For such a high pulse generation rate, signal $x(t)$ is reconstructed using the point process with relatively low error. This results in the applicability of Sauer's theorem [11] and the possibility of attractor reconstruction. Note that these results practically coincide with λ_1 calculated using the signal $x(t)$ (see figure, black circles). Dependence $\lambda_1(l)$ decreases both for large l and small l. The reason that dependence $\lambda_1(l)$ decreases for $l \geqslant 0.1$ is that this interval is beyond the domain of applicability of linear approximation. If the distance between trajectories in the phase space exceeds approximately 10% of the attractor size, the rate of trajectories divergence is no longer exponential. As a consequence, the value of λ_1 is underestimated because the perturbation vector becomes smaller than its expected value. The corresponding constraints can be approximately described as

$$\lambda_1(l) \sim \frac{1}{t'} \ln(c - dl), \tag{4}$$

where t' is the time between renormalizations and c and d are the constants ($c \gg d$). These constraints appear independently of the threshold level value, including in calculation of Lyapunov exponent using variable $x(t)$ of model (3). It can be seen from the figure that they are approximately the same in all the considered cases.

In the region of small values of l, the decreasing character of $\lambda_1(l)$ in calculations using time realization $x(t)$ is connected with the errors of vector orientation that arise in the case of very frequent renormalizations. One more factor limiting from above the estimate of the Lyapunov exponent in analysis of point processes is uncertainty δ in input signal reconstruction. If δ takes small values (as is in the considered case $\theta = 5$), λ_1 calculated using the sequence I_i is almost the same as that determined using the known realization $x(t)$ (see figure). With increasing δ, however, the value of λ_1 decreases. Let us consider for simplicity the case of a similar uncertainty for the perturbation vector before and after renormalization [18]; then, dependence $\lambda_1(l)$ can be approximately estimated as

$$\lambda_1(l) \sim \frac{1}{t'} \ln\left(\frac{l + \delta}{r + \delta}\right), \tag{5}$$

where r is the initial chosen perturbation vector. The larger δ, the more strongly λ_1 is limited from above (see figure). As a consequence, the range of l in which the largest Lyapunov exponent can be correctly determined exhibits significant narrowing with increasing θ. Thus, for $\theta = 30$ (seven pulses per characteristic oscillation period), it is noticeably smaller than for $\theta = 5$, and for $\theta = 50$ constraints (4) and (5) for λ_1 result in the fact that the interval corresponding to correct estimates of $\lambda_1 = 0.087$ is narrower by approximately a factor of 2, which illustrates the importance of choosing parameter l.

Thus, in this letter, we elucidated the constraints of the method for calculating dynamical parameters using the point process of the integrate-and-fire model. In our opinion, the calculation of the function $\lambda_1(I)$ and the maximum of this function (averaged over variation of the algorithm parameters, the delay time, and the dimension of the space of embedding) is a method for very accurate determination of the dynamical parameters of complex oscillatory regimes in reconstructing dynamical systems using point processes. This improvement of the method [10] increases the reliability of estimates for the largest Lyapunov exponent.

Acknowledgments. This study was supported by the Russian Science Foundation, project no. 14-12-00224.

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Translated by E. Baldina