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The Evolution of Spatiotemporal Chaos in a Discrete-Continuous Active Medium

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Abstract—A special approach to calculation of the spectrum of Lyapunov exponents has been developed and applied to diagnostics of the degree of regularity of wave structures in a model neural system. A transition between the regimes of regular wave dynamics and developed spatiotemporal chaos is demonstrated and quantitatively characterized.

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Knowledge that is gained regarding the functioning of the human brain inspires periodic revision of the paradigms underlying physical models of neural systems. In particular, modern notions of various processes accompanying phenomena, such as propagating cortical depression, migraine, and depolarization waves upon cerebral trauma or stroke go far beyond the framework of a model representation of the neuronal ensemble as a discrete set of active elements with certain topology of mutual links [1–4]. In the aforementioned cases, this discrete set of elements is "built" into a diffusion medium that plays the role of accumulator and transporter of substances controlling the neural activity. Mathematical models adequate to formulated problems belong to the class of mixed discrete-continuous systems. Studying the dynamics of these systems frequently requires a special approach. On a simplest level, this neural system can be described in terms of a three-component model of the reaction–diffusion type, where a partial differential equation describes the transport of substances (potassium ions, potassium glutamate) in the intercellular space, while a system of ordinary differential equations (ODEs) defined in a discrete set of positions characterizes activity of the neural ensemble [5].

The present work was aimed at a quantitative analysis of the transition from spatiotemporal chaos to a regular regime of running waves in a model system representing a discrete ensemble of FitzHugh–Nagumo neurons [6] occurring at the nodes of a two-dimensional (2D) spatial network and interacting by means of diffusion coupling described by an additional variable continuously distributed in the interaction space.

Let us consider the corresponding mathematical model. Assuming that neurons occur at points with coordinates $r_{ij} = (x_i, y_j)$, where $x_i = ihx$, $y_j = jhy$, and h is the spatial step, their dynamics can be described by the following ODEs:

$$\varepsilon_v \frac{dv_{ij}(t)}{dt} = v_{ij}(t) - v_{ij}^3(t)/3 - w_{ij}(t) + z(x_i, y_i, t),$$
$$\tau_l \frac{dw_{ij}(t)}{dt} = A + Bv_{ij}(t) - w_{ij}(t). \quad (1)$$

Here, $v_{ij}(t)$ and $w_{ij}(t)$ are the variables characterizing the electrical activity of each neuron $(i, j = \overline{1, 40})$; function $z(x_i, y_j, t)$ defined continuously in space $\mathbf{r} = (x, y)$ describes the spatiotemporal evolution of the concentration of substances in the intercellular space [5]:

$$\varepsilon_z \frac{\partial z}{\partial t} = \alpha_z \Psi(v) - z + \gamma\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right),$$
$$\Psi(v) = \frac{1}{2}\left(1 + \tanh\left(\frac{v}{v_s}\right)\right), \quad (2)$$

where $\Psi(v)$ is a logistic function dependent on parameter v and having two asymptotic limits, zero and unity. Equations (1) and (2) are supplemented with boundary conditions, those for variable z being determined by the peculiarities of a particular problem. In the present work, the active medium was modeled by a rectangular space with dimensions $L_x \times L_y$ ($L_x = 1$, $L_y = 1$) and mixed boundary conditions. In particular, the boundary in coordinate y was described by the

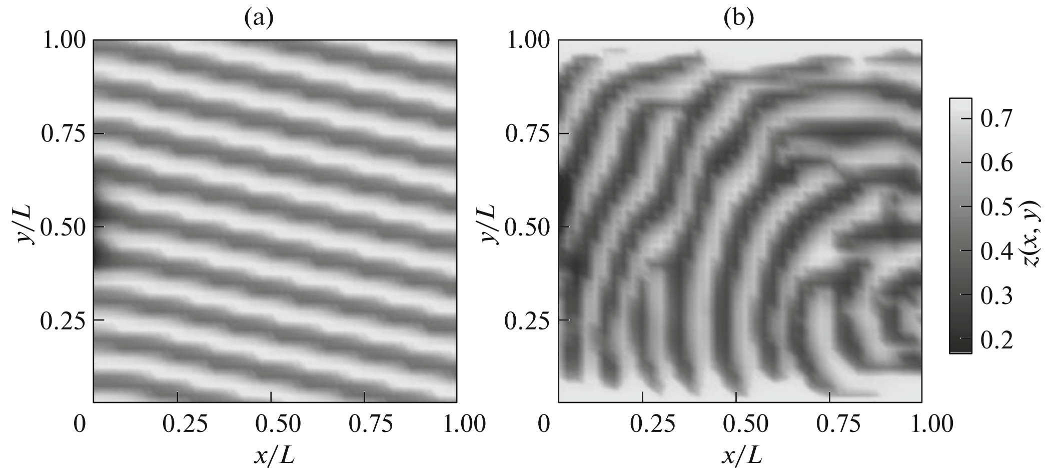

Fig. 1. Instantaneous distributions of substance concentration $z(x, y)$ in the intercellular space in the limiting cases of (a) $k_d = 1$ (connected boundaries) and (b) $k_d = 0$ (disconnected boundaries).

Neumann and Dirichlet conditions, while the boundary conditions in coordinate x were periodic:

$$\begin{aligned}\varepsilon_z &= \frac{dz_{1,j}}{dt} = \alpha_z \Psi(v_{1,j}) - z_{1,j} \\ &+ \gamma(z_{2,j} - z_{1,j}) + k_d \gamma(z_{N,j} - z_{1,j}), \\ \varepsilon_z &= \frac{dz_{N,j}}{dt} = \alpha_z \Psi(v_{N,j}) - z_{N,j} \\ &+ \gamma(z_{N-1,j} - z_{N,j}) + k_d \gamma(z_{1,j} - z_{N,j}),\end{aligned} \quad (3)$$

where $z_{1,j}$ and $z_{N,j}$ ($\forall j = 1, \ldots, N$) correspond to values of variable z on the left and right boundaries of the medium, respectively. According to [5], this spatial configuration under certain conditions provides for the appearance of an autonomous leading center.

Model equations (1)–(3) were numerically integrated on a spatial grid with step $h = 0.025$ for a system with the following parameters: $A = 0.5$, $B = 1.1$, $\tau_l = 1.0$, $\varepsilon_z = 1.0$, $\alpha_z = 1.1$, $\varepsilon_v = 0.004$, $\gamma = 7.5 \times 10^{-4}$, and $v_s = 0.05$. The control parameter was k_d, which determined the spatial configuration of the system. In the limiting case of $k_d = 1$, the left and right boundaries are connected and admit the mutual diffusion transport of substances while in the case of $k_d = 0$ the boundaries are disconnected. In intermediate cases, $0 < k_d < 1$, the control parameter provides a correction to the diffusion coefficient that describes the transport of substances between the left and right boundaries.

Figure 1 presents the results of numerical simulation of the system dynamics in the limiting cases of $k_d = 1$ (Fig. 1a) and $k_d = 0$ (Fig. 1b) in the form of "instantaneous" distributions of concentration $z(x, y)$ in the intercellular space. As can be seen, the system with $k_d = 1$ (connected boundaries) exhibits a regular structure of excitation waves propagating in one direction at equal velocities. In the case of $k_d = 0$ (disconnected boundaries), the system features non-regular dynamics that is characterized by the presence of a spatial region of chaotic dynamics (leading center) generating a set of waves propagating in various directions. Evidently, a monotonic variation of parameter k_d from unity to zero ensures transition from periodic to chaotic dynamics.

For quantitative characterization of the evolution of spatiotemporal dynamics, we have calculated the spectrum of Lyapunov exponents and traced its modification depending on the variable parameter k_d. As is known, the calculation of Lyapunov exponents provides a powerful tool for analysis of the evolution of dynamical regimes in systems of various natures. However, in the analysis of spatially distributed systems with a phase space of infinite dimensionality, the calculation of Lyapunov exponents encounters difficulties related to correct determination of the state of a system under consideration and construction of the corresponding set of disturbances [7]. For the system studied in this work, this task was additionally complicated because the dynamics of a neural ensemble was described, in addition to by the partial differential equation (2), by the system of ODEs (1). In order to overcome this difficulty, we have developed a modification of a method proposed previously [8. 9] for distributed systems. According to this, the initial state of the system under consideration was set as follows:

$$\begin{aligned}U(x,y,t) &= (v_{\overline{1,N}}(x,y,t), w_{\overline{1,N}}(x,y,t), z(x,y,t))^T, \\ v_{\overline{1,N}}(x,y,t) &= v_{\overline{1,N}}(t), \\ w_{\overline{1,N}}(x,y,t) &= w_{\overline{1,N}}(t), \quad \forall x, y \in r.\end{aligned} \quad (4)$$

Figure 2 shows temporal variation of the Lyapunov sums normalized to the length of time series in the limiting cases of $k_d = 1$ (Fig. 2a) and $k_d = 0$ (Fig. 2b) corresponding to the regimes of periodic and chaotic dynamics, respectively, illustrated in Fig. 1. As can be seen, upon expiry of the period of time $T = 4000$

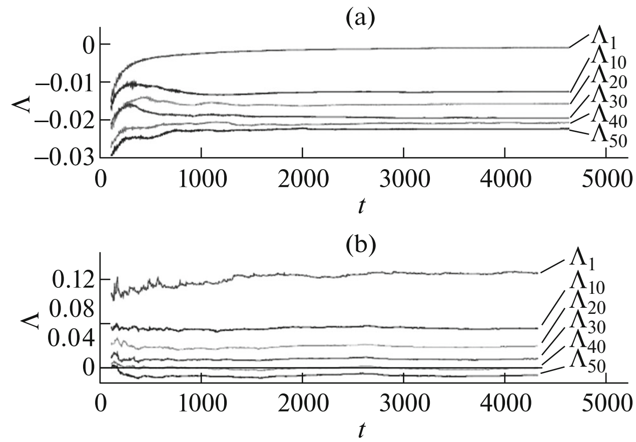

Fig. 2. Temporal variation of some maximum Lyapunov exponents in the limiting cases of (a) $k_d = 1$ and (b) $k_d = 0$.

(dimensionless units), the calculated Lyapunov exponent becomes almost stationary and no longer depends on the time and number of iterations in the Gram–Schmidt process. The regime of regular spatiotemporal dynamics (Fig. 2a) with periodic oscillations of $z(x, y)$ (Fig. 1a) is characterized by the presence of zero maximum Lyapunov exponent. On the other hand, Fig. 2b shows that the dynamical regime characterized by the formation of a leading center (Fig. 1b) exhibits a set of $N_\Lambda = 39$ positive maximum Lyapunov exponents (Fig. 2b). From the standpoint of Lyapunov exponents, this regime can be treated as hyperchaotic, in which positive exponents in the spectrum are mostly related to the infinite dimensionality of the phase space of the system under consideration.

For studying the scenario of the transition from periodic to hyperchaotic dynamics, we have calculated the dependence of fifty largest Lyapunov exponents[1] on the value of control parameter k_d (Fig. 3). As can be seen from this dependence, the region of $0.58 < k_d < 1.0$ corresponds to a periodic regime with a zero maximum Lyapunov exponent, while $0 < k_d < 0.26$ corresponds to a chaotic regime with $N_\Lambda = 39$ positive maximum Lyapunov exponents. It should be noted that the variation of k_d in these intervals is not accompanied by the complication of dynamics, and the system features typical spatiotemporal structures analogous to those presented in Fig. 1. At the same time, variation of the control parameter within $0.26 < k_d < 0.58$ is accompanied by numerous transitions between regular and irregular wave regimes.

In concluding, we have discovered and quantitatively characterized transitions between various types of wave structures in a generalized model of a neural medium. It was established that a change in the type of boundary conditions and region remote from the leading center (wave source) can decisively influence the resulting spatiotemporal dynamics. A specially modified algorithm of calculation of the spectrum of Lyapunov exponents for the model discrete-continuous medium allowed evolution of the observed wave regimes to be quantitatively characterized and studied in detail. The results presented in this Letter open up at least two promising directions of research, one of which is related to analysis of the mechanisms of generation of self-sustained structures and waves in systems studied, while the other is related to further development of the method of Lyapunov exponents in application to discrete-continuous active media.

[1]With allowance for the infinite dimensionality of phase space of the system under consideration, hyperchaotic regimes can be characterized by a large ($N \sim 40$) number of positive Lyapunov exponents.

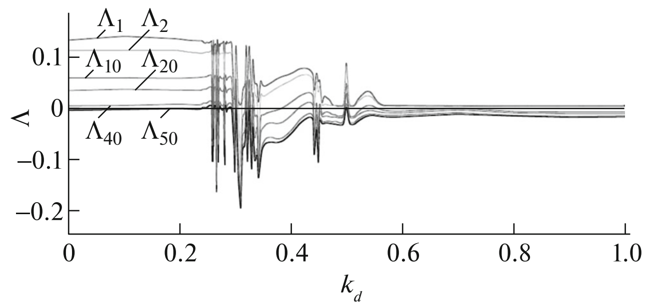

Fig. 3. Largest Lyapunov exponents calculated for various values of control parameter k_d.

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