Multilayer structure formation via homophily and homeostasis

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ABSTRACT

The competition of homophily and homeostasis mechanisms taking place in the multilayer network where several layers of connection topologies are simultaneously present as well as the interaction between layers is considered. We have shown that the competition of homophily and homeostasis leads in such networks to the formation of synchronous patterns within the different layers of the network, which may be both the distinct and identical.

Keywords: multiplex networks, homophily, homeostasis, Kuramoto oscillators, synchronization, cluster formation

1. INTRODUCTION

Coupled biological and chemical systems, social groups and interacting animal species, the Internet and the World Wide Web, the brain and the stock markets are just a few examples of systems composed of a huge number of highly interconnected dynamical components. The modern approach to capture the global properties of such systems is to model them as graphs,[1–3] where nodes represent the basic units, and links stand for the interactions between them, forming a specific connectivity pattern which defines the so-called network's topology. Despite their intrinsic differences, a set of surprising common properties (such as a power law scaling in the network connectivity and a modularity structure observed at the mesoscopic scale) has been revealed in real-world network (RWN) structures.[4] The spontaneous emergence, as a self organization process, of these topological features has been explained recently[5,6] as the direct consequence of structure-dynamics adaptation principles involving the competition between two basic mechanisms. The first one (well established in sociology and neuroscience under the terms of *homophily* [7] and Hebbian learning, respectively) corresponds to the trend of reinforcing those interactions with other correlated units in the graph. The second (which preserves the value

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of the input strength received by each unit) results from the limitation in the associative capacity, and is known to play a relevant role in neuroscience under the term *homeostasis* ,[8] while in social systems it is related to the so called Dunbar's number.[9]

Only in the last years, and taking advantage of an enhanced resolution in real data sets, the interest switched to properly frame the *multilayer* character of RWNs, by considering them as networks made, in fact, of diverse relationships (layers) between its constituents.[10] In this Report we show that competition between homophily and homeostasis actually leads to self-organization of ensembles of oscillators into a multilayer network structure.

2. MODEL

We consider an ensemble of N oscillators, where each oscillator i ($i = 1, ..., N$) has it's eigenfrequency ω_i, and described by a vector of M components ϕ_i^l ($l = 1, ..., M$) corresponding to it's time dependent phase in each of the M layers of the network on which it interacts with the rest of the ensemble. Here, a Kuramoto-like evolution for the phase $\phi_i^l(t)$ on each layer $l = 1, ..., M$, is assumed. Also, each oscillator interacts with itself within it's phase vector $\vec{\phi}_i$, that presents the interaction between layers. The resulting evolution of the phase vectors is given by

$$\dot{\phi}_i^l(t) = \omega_i + \lambda_1 \sum_{j \neq i} w_{ij}^l(t) \sin(\phi_j^l - \phi_i^l) + \lambda_2 \sum_{j \neq l} \sin(\phi_i^j - \phi_i^l). \tag{1}$$

Here, $\{\omega_i\}$ is a set of randomly assigned natural frequencies distributed uniformly in $[-\pi, \pi]$ (note that the natural frequency ω_i of i-th oscillator is the same for all M layers of the network), λ_1 and λ_2 are the intra-layer and inter-layer coupling strengths, respectively. Finally, $w_{ij}^l(t)$ is the weight of the connection between elements i and j on layer l (whose time evolution will be momentarily described). On each layer l, for each oscillator i and at each time t, the set $\{w_{ij}^l\}$ satisfies the condition

$$\sum_{j \neq i}^{N} w_{ij}^l = 1. \tag{2}$$

In other words, we consider the case for which, in Eq. (2), the value of the input strength received by each unit within each layer is a constant.

In parallel with the node dynamics (1), the weights of the links are also evolving through dynamical equations that reflect competition mechanisms between homophily and homeostasis.[5,6] The adaptive evolution of the weights w_{ij}^l is governed by

$$\dot{w}_{ij}^l(t) = p_{ij}^l(t) - \left(\sum_{k \neq i} p_{ik}^l(t) \right) w_{ij}^l(t), \tag{3}$$

where the time dependent quantity $p_{ij}^l(t)$ is defined as

$$p_{ij}^l(t) = \frac{1}{T} \left| \int_{t-T}^{t} e^{i(\phi_i^l(t') - \phi_j^l(t'))} dt' \right|. \tag{4}$$

Notice that p_{ij}^l denotes, at time t, the average phase correlation (within layer l) between oscillators i and j over a characteristic memory time T. It follows from Eq. (3) that the normalization condition (2) holds at all times, i.e., the sum of the weights of all incoming connections at each node within each layer is conserved.

The case of a monoplex ($M = 1$) was extensively studied in Refs. ,[5,6] both numerically and analytically, and it was shown that a large region exists in the parameter space (σ_1, T) where, starting from random initial conditions for the weights w_{ij}^1 and from random initial phases ϕ_i^1 in interval $[-\pi, \pi]$, Eqs. (1,3) asymptotically lead to the spontaneous accommodation of the ensemble into a cluster-synchronized regime.

As we are, instead, focused to investigate the emergence of a multilayer network structure, in the following we will fix $T = 100$, and concentrate on the analysis of the solution of Eqs. (1,3) for $N = 100$ oscillators and $M = 10$ layers of connections, as a function of inter-layer coupling parameter λ_2.

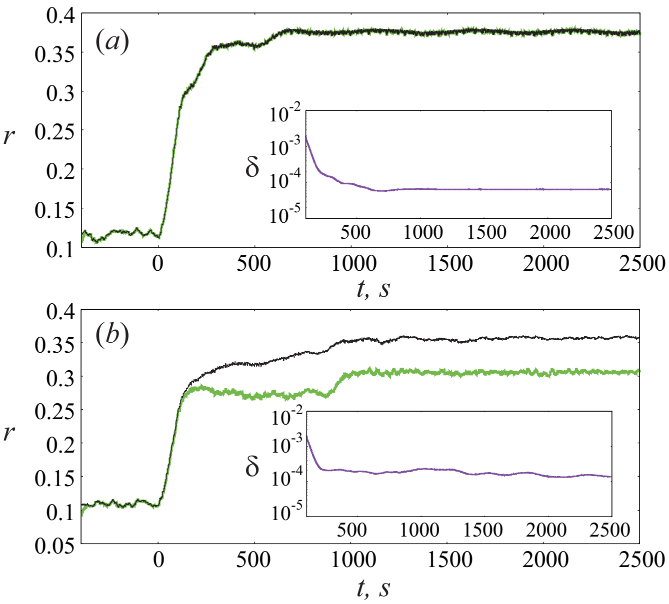

Figure 1. Global (grey line) and intra-layer (black line) order parameter for different values of inter-layer coupling strength: (a) $\sigma_2 = 0.012$ and (b) $\sigma_2 = 0.012$. Intra-layer coupling parameter is $\sigma_1 = 1$. The insets shows the evolution of δ value (see definition in text).

3. RESULTS

Here, we are interested in possibility to observe the multilayer structure as a result of adaptive interactions via homophily and homeostasis. To detect it we compare the classical time-dependent order parameter, $r(t)$, reflecting the global phase coherence used in[5] with the phase coherence inside each layer of our model, averaged over all system:

$$r_{layers}(t) = \frac{1}{M}\sum_{k=1}^{M} r^k(t) = \frac{1}{MN}\sum_{k=1}^{M}\left|\sum_{j=1}^{N} e^{i\phi_j^k(t)}\right|. \quad (5)$$

Here, the value of $r(t)$ lower then $r_{layers}(t)$ will reflect the incoherence between the same node in different layers, that, by turn, will stand for the presence of the multilayer structure.

This comparison is shown in the Fig. 1 for two different values of inter-layer coupling strength $\sigma_2 = 0.02$ (Fig.1(a)) and $\sigma_2 = 0.012$ (Fig.1(b)). The value of σ_1 is chosen according to the study of the monoplex system.[5,6] Firstly, for $t < 0$ we integrate numerically the equation (1) in a homogeniuos network, where nodes are connected all-to-all with randomly assigned weights, which are constant in time, and the condition (2) satisfied. Then, for $t \geq 0$ we considered the full dynamics of the adaptive model by switching on the weight evolution presented by equation (3). To detect that the weights become constant and the structure has reached the steady state we calculate the quantity $\delta = \frac{1}{M}\sum_l^M \sqrt{\sum_{i,j}[W_{ij}^l(t) - W_{ij}^l(t-1)]^2}$, which is depicted in insets of the figures. This value is decreasing before $t \approx 500$, then becomes almost constant and have magnitude $10^{-3} - 10^{-5}$, that is small enough to consider the weights are not changing and structure is stationary.

Then the weights are constant in time ($t < 0$) the dynamics is similar in both cases: the values of $r(t)$ and r_{layers} are equal to each other, and their magnitude is about 0.1, i.e. the system acts like 1-layer random network. Then the adaptation is turned ON ($t = 0$) via the equation (3), the magnitude of both order parameters increases dramatically. Furthermore, at $t \approx 100$ the crucial difference appears between case (a) and (b). Considering the case (a), one can see the values of $r(t)$ and $r_{layers}(t)$ remain equal until the structure of the network has reached the steady state. In contrast, in case (b) the global order parameter, $r(t)$, departs from intra-layer order, $r_{layers}(t)$, and becomes sufficiently lower.

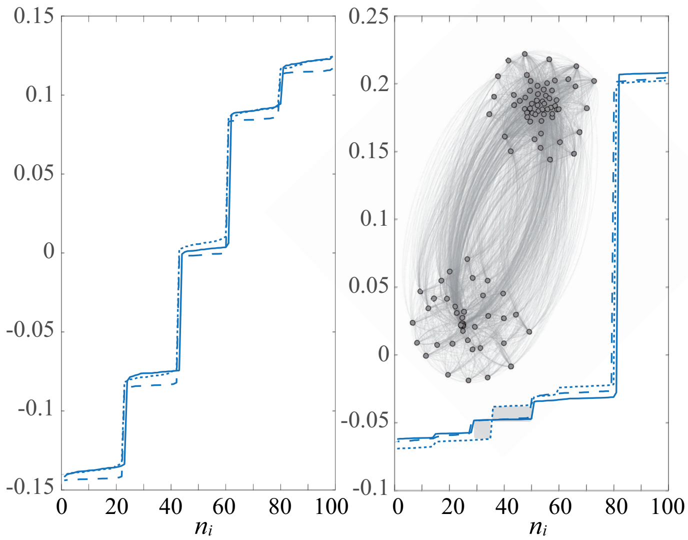

Figure 2. Values of second lowest eigenvector of the first (solid line), second (dotted line) and third (dashed line) layer sorted by it's magnitude. Plot (a) presents evaluation of the model while $\sigma_2 = 0.012$ and (b) corresponds to $\sigma_2 = 0.012$. Intra-layer coupling parameter is $\sigma_1 = 1$. The inset in plot (b) reflects the visualization of the difference matrix between first and second layer, while $\sigma_2 = 0.012$.

To study this effect in detail we investigate the structure of the network using the second smallest eigenvectors of layer matrixes, also called Fiedler vectors,[11] associated with algebraic connectivity[11] of corresponding nodes. To catch the partition inside each layer, we sort the values of corresponding Fiedler vector according to it's magnitude. For clear representation only three layers are depicted in Fig. 2. Then $\sigma_2 = 0.02$ (Fig.2(a)) one can see the stair-like curves, where each level represent itself the strongly coupled community. Five levels can be observed on each curve, moreover, the curves are almost identical and the levels are completely overlap. The latter reflects the same community structure within layers.

The structure becomes more complex, while we decrease the inter-layer coupling strength, plotted in Fig. 2(b). Most of communities still identical at all layers, in particular, the one with highest connectivity. However, there is a considerable non-overlapping region (marked by grey) between second and first (or third by reason of similarity) layer. Namely, one community has merged with it's neighbors within second layer in contrast with other layers, that has led to emergence of multilayer structure.

To gain deeper insight in topology we calculate the absolute difference matrix between first and second layer as $w_{ij}^d = |w_{ij}^1 - w_{ij}^2|$. This matrix was visualized with help of Gephi,[12] OpenOrd layout and shown in Fig. 2(a). For the sake of simplicity we neglect the weights lower than 10^{-3}. The picture reflects two strongly-coupled communities. Here, links between clusters represent themselves the vanished connections between nodes of the community that has merged with it's neighbors in the second layer. In turn, links inside depicted clusters stand for connections, that are missed in the first layer.

4. CONCLUSIONS

In conclusion, we have considered the competition of homophily and homeostasis processes in the multiplex network. We have shown that the competition of homophily and homeostasis results in the formation of synchronous patterns which may be both the distinct and identical within the different layers of the complex network.

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