Generalized synchronization in the complex network: theory and applications to epileptic brain

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ABSTRACT

Generalized synchronization in complex networks with chaotic dynamical systems being in their nodes has been studied. The synchronous regime is shown to be detected by the sign-change of the second positive Lyapunov exponent of the network or by the nearest neighbor method. The same method is shown to be applied for the detection of the synchronous regime between the different fields of epileptic brain.

Keywords: Generalized synchronization, Lyapunov exponents, nearest neighbors, neurophysiological data, epileptic brain

1. INTRODUCTION

Chaotic synchronization of coupled dynamical systems is one of the relevant directions of nonlinear science.[1] In the simplest case chaotic synchronization can be observed in two coupled dynamical systems. Last time the special attention of investigators is paid to the analysis of interaction of a large number of nonlinear systems, i.e. networks of coupled nonlinear elements.[2] At that, as a main subject of investigations the complex networks having non–regular structure and high heterogeneity of couplings with chaotic dynamical systems being in their nodes can be considered.[3] The interest to such systems is connected both with the necessity of analysis of systems of different kind of nature including the neurophysiological ones and the significance of investigation of fundamental aspects of chaotic synchronization in complex networks.

Several types of chaotic synchronization in complex networks are traditionally distinguished. They are complete synchronization,[4,5] phase synchronization,[3] generalized synchronization,[6,7] cluster synchronization[8] and others. At that, complete, phase and cluster synchronization in complex networks are well studied recently, whereas analysis of generalized synchronization in such systems has not been performed in detail so far. As an exception one can mention paper[7] where generalized synchronization in unidirectionally coupled complex networks has been considered. At that, the most general case of complex network configuration has not been studied in detail. Known works are directed to the finding of the fact of the generalized synchronization regime existence whereas the theory of generalized synchronization regime has not been yet proposed. Moreover, in all known works the detection of generalized synchronization has been performed by the modification of auxiliary system approach[9] being the effective tool for the analysis of generalized synchronization in systems with unidirectional coupling whereas such modification even for two mutually coupled dynamical systems is misleading.[10]

In present paper we describe the concept of generalized synchronization in complex networks. We show that generalized synchronization in complex networks can be detected by calculation of the spectrum of Lyapunov exponents for the whole network or by the nearest neighbor method. At that, known cases of two unidirectionally or mutually coupled dynamical systems[11,12] can be considered as partial cases of the developed concept.

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2. THEORY OF GENERALIZED SYNCHRONIZATION IN COMPLEX NETWORKS

First of all, let us properly set up a definition for the generalized synchronization regime in complex networks. As in the case of two unidirectionally or mutually coupled dynamical systems, the generalized synchronization in complex networks means the presence of functional relation between the interacting system states which for the generic ensemble of N networking elements would be given by

$$\mathbf{F}[\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_i(t), \ldots, \mathbf{x}_N(t)] = 0, \tag{1}$$

where $\mathbf{x}_i(t) = (x_i, y_i, z_i)^T$ is the vector state of the i-th element of the network. To reveal the mechanisms resulting in the generalized synchronization regime onset in the network we use the modified systems approach[13] proposed by us previously for two unidirectionally coupled dynamical systems. For the purpose of exemplification, and without lack of generality, we here-below characterize the state of the network by the only vector $\mathbf{U} = (u_1, u_2, \ldots, u_i, \ldots, u_{N \cdot N_d})^T$, where $u_{3i-2} = x_i$, $u_{3i-1} = y_i$, $u_{3i} = z_i$, instead of the set of vectors $\mathbf{x}_i$, $i = \overline{1, N}$. Notice that, here, the dimension of each network's element is assumed to be $N_d = 3$, but this analytical study may be extended easily to the other systems with arbitrary dimensions N_d. Following the above formalism, the entire network may be considered as a high-dimensional autonomous dynamical system, whose evolution equation is given by

$$\dot{\mathbf{U}} = \mathbf{L}(\mathbf{U}) + \varepsilon \mathbf{G} \mathbf{U}. \tag{2}$$

Here the vector function $\mathbf{L}(\cdot)$ determines the evolution of the node elements in the absence of coupling, whereas the additive term $\varepsilon \mathbf{G}$ describes the influence of the topology and the coupling strength of the links between oscillators. Matrix $\mathbf{G}$ specifies the structure of the dissipative couplings between nodes, and it is assumed to be a symmetric zero row sum matrix, $G_{ii} = -\sum_{j \neq i} G_{ij}$, with G_{ij} ($i \neq j$) being equal to 1 whenever variable u_i forces the variable u_j and 0 otherwise.

It is easy to see that the term $\varepsilon \mathbf{G} \mathbf{U}$ brings the additional dissipation into the system (2). Indeed, the phase flow contraction is characterized by means of the vector field divergence

$$\lim_{\Delta t \to 0} \lim_{\Delta V \to 0} \frac{1}{\Delta V} \frac{\Delta V}{\Delta t} = \operatorname{div} \mathbf{L} + \varepsilon \sum_{i=1}^{N_d N} G_{ii}, \tag{3}$$

where ΔV is the elementary volume of the phase space of the system (2). Since $G_{ii} \leq 0$, the term $\varepsilon \sum_{i=1}^{N_d N} G_{ii}$ is also negative and the dissipation in the considered network increases with the growth of the coupling strength ε, resulting in the simplification of the otherwise chaotic dynamics of the system (2).

To characterize the complexity of the motion, the spectrum of Lyapunov exponents is frequently used.[14–17] In the case under study, let us suppose that the behavior of the system (2) is initially described by the set $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N N_d}$ of Lyapunov exponents, with N of them being positive. As dissipation increases, some of the initially positive Lyapunov exponents become negative, and each passage of a Lyapunov exponent trough zero testifies that one more degree of freedom of the chaotic motion corresponds to a contractive direction. When λ_2 becomes negative, only one degree of freedom is representative of the evolution of the network, i.e. a GS regime is built. Indeed, as soon as λ_2 is negative, all systems have to arrange their evolution into a specific collective motion, wherein the functional relation (1) is taking place. Note also, that the negativity of the second Lyapunov exponent λ_2 as a criterion of the GS existence coincides with the analogous one used usually for two unidirectionally coupled chaotic oscillators.[11,13]

3. GENERALIZED SYNCHRONIZATION IN NETWORKS OF REFERENCE DYNAMICAL SYSTEMS

To illustrate the proposed approach we consider the generalized synchronization regime onset in networks consisting of $N = 5$ Rössler systems with slightly mismatched control parameter values. For the simplicity of consideration the topology of links between nodes has been selected in such a way that each element of the
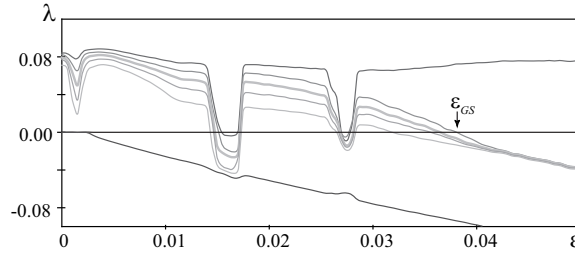
Figure 1. Dependencies of seven highest Lyapunov exponents on the coupling parameter ε for the network of five Rössler systems (4). The onset of the GS regime in the network ε_{GS} is shown by arrow

network has been connected with each other. The dynamics of i-th element of the network $(i = 1, \ldots, N)$ is described by the following ordinary differential equations

$$
\begin{aligned}
\dot{x}_i &= -\omega_i y_i - z_i + \varepsilon \sum_{j=1}^{N} C_{ij} x_j, \\
\dot{y}_i &= \omega_i x_i + a y_i, \\
\dot{z}_i &= p + z_i(x_i - r),
\end{aligned}
\tag{4}
$$

where $a = 0.15$, $p = 0.2$, $r = 10$, $\omega_1 = 0.95$, $\omega_2 = 0.9525$, $\omega_3 = 0.955$, $\omega_4 = 0.9575$, $\omega_5 = 0.96$ are the control parameter values, $\mathbf{x}_i(t) = (x_i, y_i, z_i)^T$ is the vector-state of the i-th element, ε is a coupling parameter, C_{ij} is the element of the coupling matrix

$$
\mathbf{C} = \begin{pmatrix}
-4 & 1 & 1 & 1 & 1 \\
1 & -4 & 1 & 1 & 1 \\
1 & 1 & -4 & 1 & 1 \\
1 & 1 & 1 & -4 & 1 \\
1 & 1 & 1 & 1 & -4
\end{pmatrix}
\tag{5}
$$

of the network.

Dynamics of the network (4) is characterized by the spectrum of $3N = 15$ Lyapunov exponents. Due to the presence of chaotic dynamics in each element, in the absence of coupling the considered network contains N positive, N negative, and N zero Lyapunov exponents. With the coupling parameter value increasing initially zero and positive Lyapunov exponents pass sequentially in the field of the negative values. In Fig. 1 the dependencies of seven highest Lyapunov exponents of considered network on the coupling parameter ε have been shown. It is clearly seen that at $\varepsilon_{GS} \approx 0.0385$ the second Lyapunov exponent λ_2 passes through zero and becomes negative. Therefore, the generalized synchronization regime should be observed for the coupling parameter values $\varepsilon > \varepsilon_{GS}$.

To confirm the presence of generalized synchronization in the network of coupled Rössler systems we have applied the nearest neighbor method[18, 19] to the system under study. The main idea of such method consists in the fact that the presence of the functional relation between the interacting system states means that all close states in the phase space of one given system should correspond to the close states in the phase spaces of all other systems. Fig. 2 illustrates the behavior of nearest neighbors and their images in the phase space of all Rössler systems of the network for two different values of the coupling parameter. Fig. 2, a corresponds to the small enough value of the coupling parameter ($\varepsilon = 0.03$) whereas Fig. 2, b refers to the value of coupling parameter strength ($\varepsilon = 0.04$) being above the critical point ε_{GS}.

In the phase portraits of three systems $\mathbf{x}_i(t)$, $i = 2 \div 4$ three points (one point for each system) with its nearest neighbors have been selected randomly and corresponding to them points have been detected in all other systems of the network. It is clearly seen that for $\varepsilon = 0.03$ (Fig. 2, a) the points are concentrated in a limited range of attractor and distributed along the radius being the evidence of the presence of the phase and absence of generalized synchronization regimes. For $\varepsilon > \varepsilon_{GS}$ (Fig. 2, b) all states of all Rössler systems are nearest neighbors, proving the existence of generalized synchronization. So, it is possible to detect the presence of generalized synchronization in complex networks both by calculation of the spectrum of Lyapunov exponents and the nearest neighbor method.
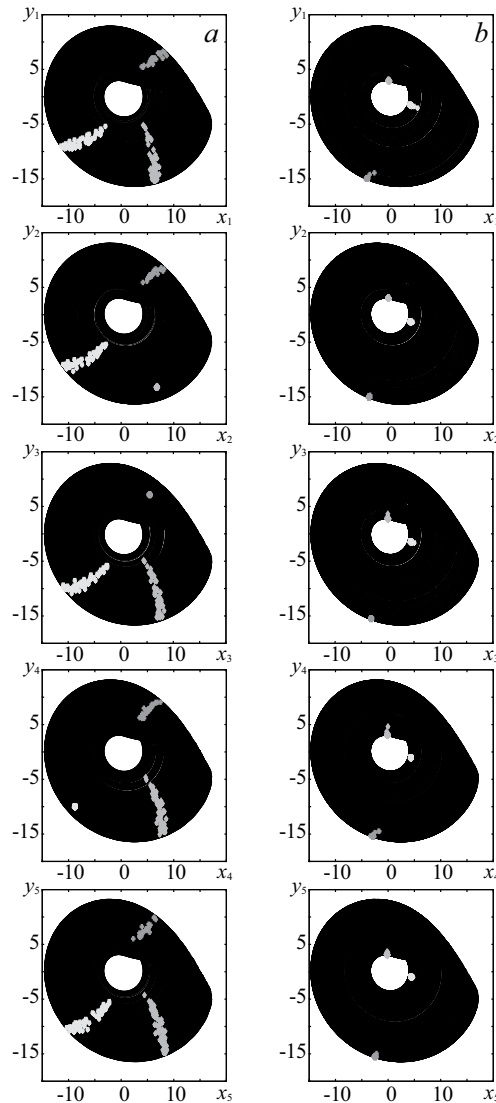
Figure 2. Behavior of nearest neighbors and their images in the phase space of the network of five Rössler systems (a) bellow ($\varepsilon = 0.03$) and (b) after ($\varepsilon = 0.04$) the generalized synchronization regime

4. GENERALIZED SYNCHRONIZATION IN EPILEPTIC BRAIN

It should be noted that the nearest neighbor method is an universal technique allowing to detect the presence of the synchronous regime by time series of interacting systems independently on the way of their obtainment. Therefore such approach can be easily applied to experimental time series. As such signals we analyze the neurophysiological data represented by the signals of electroencephalograms taken down different areas of the human brain cortex.[20]

Electroencephalograms are the complex noisy signals.[21–23] Therefore to apply the nearest neighbor method to such time series we should first of all reconstruct the system attractor by the time series with the filtered noise. For such purposes we have used Fourier transform, i.e. we have calculated the Fourier spectrum of the analyzed signals, defined the required frequency range for which the inverse Fourier transform has been then performed.[24] The obtained in such a way signal would not contain the parasitic noise that makes the reconstruction of the system attractor possible, for example, by means of the Takens method.
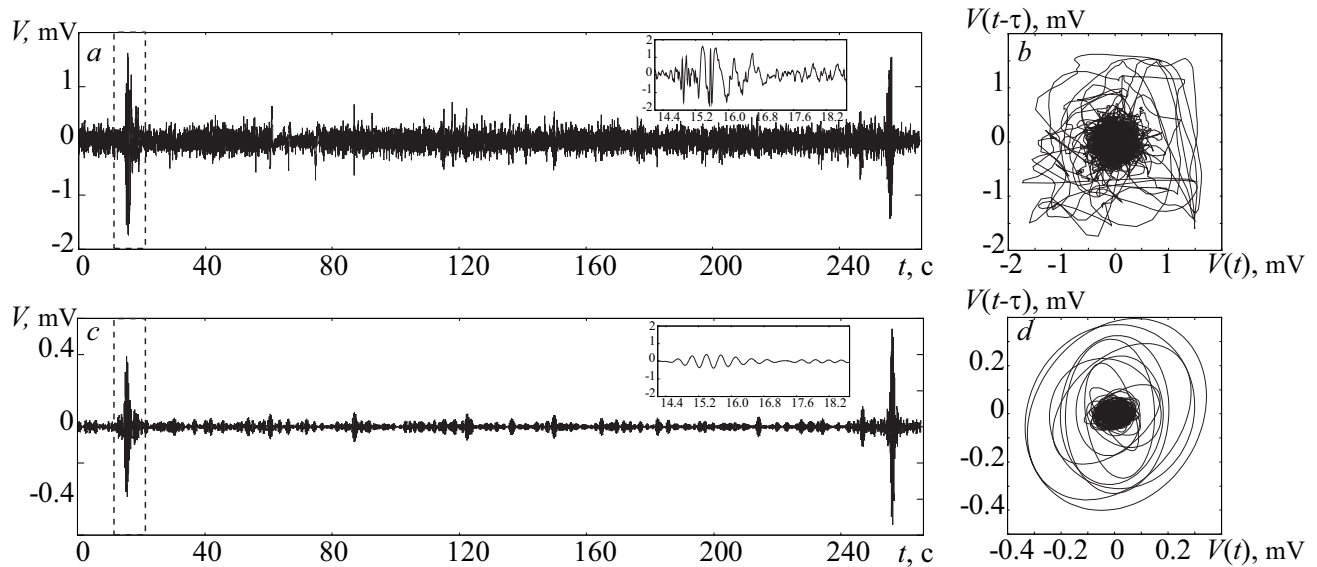
Figure 3. Fragments of electroencephalograms of human brain (a,c) and attractors reconstructed from such signals (b,d) before (a,b) and after (c,d) filtration in the frequency range $f \in 2.5 - 3.5$ Hz, $\tau = 0.08$ s
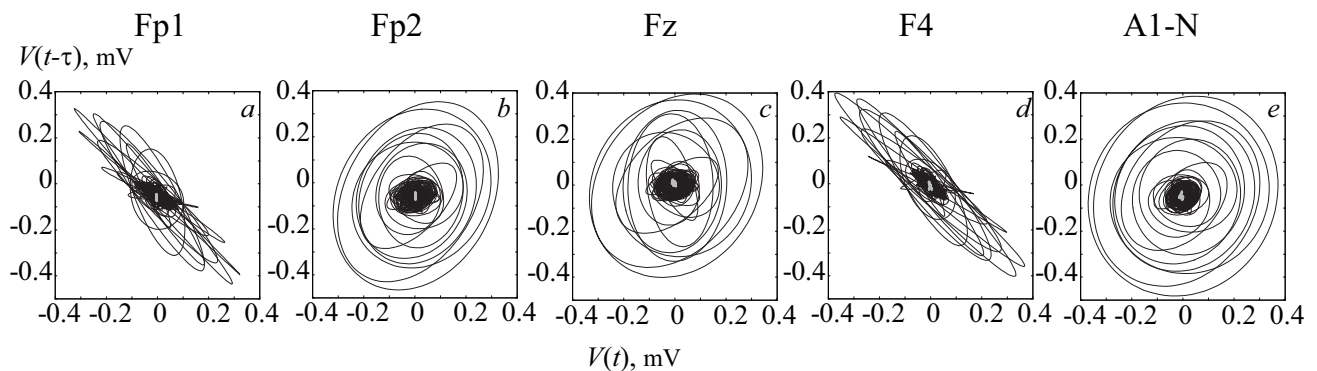

Figure 4. Attractors reconstructed from the fragments of electroencephalograms of five different areas of human brain, $\tau = 0.08$ s. The reference states (or images of reference states) and their nearest neighbors (images of nearest neighbors) are marked by points in each attractor

In Fig. 3 the original (a) and filtered (c) (in the frequency range $f \in 2.5 - 3.5$ Hz) signals of electroencephalogram and reconstructed from them attractors (b,d) are shown. It is clearly seen that due to the high noisiness of the time series the reconstructed attractor in the first case (b) looks like a "smeared spot" that is typical for the stochastic processes. At the same time, filtration of the signal in required frequency range allows to see the fine structure of the attractor (compare Fig. 3 b and d) that allows to apply the nearest neighbor method to such time series.

Let us select randomly the reference point in the filtered signal of electroencephalogram corresponding to the one area of epileptic brain and analyze the location of the images of nearest neighbor in other signals. In Fig. 4 the reconstructed attractors corresponding to five different fields of epileptic brain are shown. It is clearly seen that in all considered cases the images of nearest neighbors are close to each other testifying the presence of generalized synchronization between different areas of epileptic brain.

So, it is possible to observe the presence of generalized synchronization between the different areas of the human brain by means of the nearest neighbor method. We assume that such method can be important for the analysis of direction of couplings between different areas of brain network.

5. CONCLUSIONS

In present paper we propose the concept of generalized synchronization in complex networks with the chaotic dynamical systems being in their nodes. We show that the synchronous regime onset can be detected by the sign-change of one of the positive Lyapunov exponents taking place with the coupling parameter value increasing. We verify the presence of generalized synchronization by means of the nearest neighbor method. We show that the same method can be applied for the detection of the synchronous regime by experimental neurophysiological data.

ACKNOWLEDGMENTS

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